Utilization of Smoothing Formulae for Describing Hydraulic Relationships in Water Networks.

J. Hartley
A. Bargiela

Department of Computing,
The Nottingham Trent University,
Burton Street,
Nottingham NG1 4BU, U.K.

1. Introduction

Operational control of any water distribution network requires that a reliable picture of the present operating state of the system is available. This state is uniquely determined by nodal pressures which, in turn, determine flows through network elements and indicate the amount of water in storage reservoirs. The state vector of pressures needs, however, to be interpreted in the context of the static information about hydraulic elements comprising the network, such as pipes, valves, pumps etc.. For some elements, such as pipes, calculation of flows from the information on pressure drop and static parameters is simple, but for some others, like pressure reducing valves, which exhibit strongly nonlinear and non-differentiable pressure-flow relationships, it is more difficult. A simple solution of measuring all flows and pressures is clearly not a practical proposition, so consequently, any advanced operational control of water distribution network needs to rely on mathematical modelling of the system. The role of the mathematical modelling will then be to relate a limited number of real measurements (and pseudomeasurements representing consumption estimates) through the pressure-flow relationships of network elements and to derive a consistent state vector. There is an important issue of deciding which are the most informative measurements, that need to be taken in the physical system. Techniques have been developed to relate the position and the accuracy of the meter to the benefit that can be derived from this measurement in the process of mathematical modelling [1]. The focus of this paper is the mathematical modelling itself.

No two water networks are identical but for the purpose of network modelling one can disregard detail and focus on their essential features. The aim of water distribution networks is to transport water from one or more sources to consumers distributed over a certain area. The networks

are made up of interconnected pipes, pumps and valves. For each pipe, its length, diameter and roughness must be known, alongside the operational parameters of the pumps and valves. Each of these different types of link in the network can be modelled mathematically, by deriving hydraulic equations which relate the water flow rate to the pressure difference along the element.
The hydraulic elements which have been considered are:

The pipe.

$$Q_{ij} = \begin{cases} [0.27746 \times (C_{HW}) \times D^{-.54} \times L^{2.63}] \times (H_i - H_j)^{.54} & H_j < H_i \\ -[0.27746 \times (C_{HW}) \times D^{-.54} \times L^{2.63}] \times (H_j - H_i)^{.54} & H_i < H_j \end{cases} \quad (1)$$

The non-return valve.

$$Q_{ij} = \begin{cases} [0.27746 \times (C_{HW}) \times D^{-.54} \times L^{2.63}] \times (H_i - H_j)^{.54} & H_j < H_i \\ 0 & H_i \leq H_j \end{cases} \quad (2)$$

The non-return valve, second type.

$$Q_{ij} = \begin{cases} [0.27746 \times (C_{HW}) \times D^{-.54} \times L^{2.63}] \times (H_i - H_j)^{.54} & P < H_j < H_i \\ 0 & H_i \leq H_j \\ 0 & H_j < P \end{cases} \quad (3)$$

The pressure-reducing valve.

$$Q_{ij} = \begin{cases} [0.27746 \times (C_{HW}) \times D^{-.54} \times L^{2.63}] \times (H_i - H_j)^{.54} & H_j < H_i < P \\ [0.27746 \times (C_{HW}) \times D^{-.54} \times L^{2.63}] \times (P - H_j)^{.54} & H_j < P < H_i \\ 0 & H_i \leq H_j \\ 0 & P \leq H_j \end{cases} \quad (4)$$

The pressure-reducing valve, second type.

$$Q_{ij} = \begin{cases} [0.27746\times(C_{HW})\times D^{-.54}\times L^{2.63}]\times(H_i - H_j)^{.54} & H_j<H_i<P \\ [0.27746\times(C_{HW})\times D^{-.54}\times L^{2.63}]\times(P - H_j)^{.54} & H_j<P<H_i \\ -[0.27746\times(C_{HW})\times D^{-.54}\times L^{2.63}]\times(H_j - H_i)^{.54} & H_i\leq H_j<P \\ 0 & P\leq H_j \end{cases} \qquad (5)$$

The fixed-head pump.

$$Q_{ij} = \begin{cases} [0.27746\times(C_{HW})\times D^{-.54}\times L^{2.63}]\times(H_i + P - H_j)^{.54} & H_j<H_i+P \\ 0 & H_i+P\leq H_j \end{cases} \qquad (6)$$

The flow-reducing valve.

$$Q_{ij} = \begin{cases} [0.27746\times(C_{HW})\times D^{-.54}\times L^{2.63}]\times(H_i - H_j)^{.54} & P<H_j<H_i \\ [0.27746\times(C_{HW})\times D^{-.54}\times L^{2.63}]\times(H_i - P)^{.54} & H_j<P<H_i \\ 0 & H_i\leq H_j \\ 0 & H_i\leq P \end{cases} \qquad (7)$$

The flow-reducing valve, second type.

$$Q_{ij} = \begin{cases} [0.27746\times(C_{HW})\times D^{-.54}\times L^{2.63}]\times(H_i - H_j)^{.54} & P<H_j<H_i \\ [0.27746\times(C_{HW})\times D^{-.54}\times L^{2.63}]\times(H_i - P)^{.54} & H_j<P<H_i \\ [0.27746\times(C_{HW})\times D^{-.54}\times L^{2.63}]\times(H_j - H_i)^{.54} & H_i\leq H_j \& P\leq H_j \\ 0 & H_i\leq P \& H_j<P \end{cases} \qquad (8)$$

The altitude-control valve.

$$Q_{ij} = \begin{cases} [0.27746\times(C_{HW})\times D^{-.54}\times L^{2.63}]\times(H_i - P)^{.54} & H_j<P<H_i \\ 0 & H_i\leq H_j \\ 0 & H_i\leq P \\ 0 & P\leq H_j \end{cases} \qquad (9)$$

where Q_{ij} is the flow from node i to node j,
C_{HW} is the Hazen-Williams coefficient of the link,
D is the diameter of the link,
L is the length of the link,
H_i is the head in the i-th node,
H_j is the head in the j-th node, and
P is the pressure limit.

All information concerning the consumption and elevation of each node is also required. If, for the purpose of modelling, the link will denote a physical hydraulic element such as a pipe, valve or pump and the node will denote either a physical connection of links or the inlet or outlet point, (this includes reservoirs and boreholes), then the water distribution system can be represented as in Figure 1.

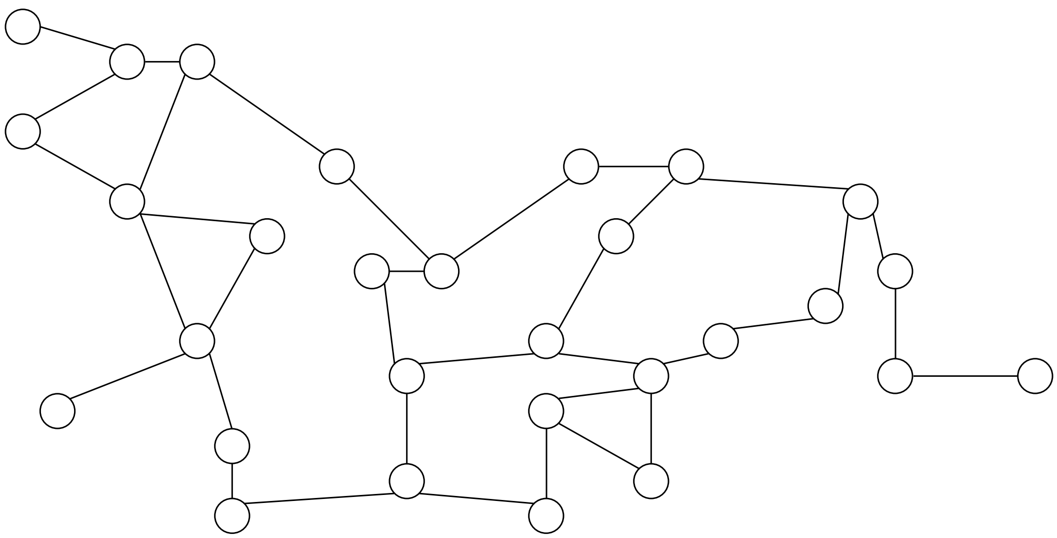


Figure 1. A network of linked nodes

○ node (physical connection, inlet or outlet)

— link (pipe, valve or pump)

The pressure-flow relationships for individual hydraulic elements are interrelated by considering the physical laws regulating the system - the flow conservation law and the loop head loss law. The flow conservation law ensures that the total flow entering a node is equal to the total flow leaving that node, while the loop head loss law states that the sum of the pressure drop around a loop of the network is equal to zero. The network equations relate the network's nodal pressures to measurement or pseudomeasurement values, through mass balance equations, and are expressed by the network equation

$$\mathbf{g}(\mathbf{x}) = \sum_{i} Q_{ij} = \mathbf{z} \tag{10}$$

x is a vector of n state variables, called the state vector, which consists of all the nodal pressures and the inflows into the fixed-head nodes. Q_{ij} is the flow from node i to node j. z is the measurement vector which consists of real measurement values and pseudomeasurements, such as predictions of nodal consumption. **g**(**x**) is the network function, which includes information about the connectivity of the network and the parameters of the pipes, pumps and valves. Since **g**(**x**) ia a nonlinear function, a direct solution is not possible, instead an iterative solution technique must be used.

The Newton-Raphson method (an iterative procedure) is used. Primarily **g**(**x**) is expanded by an initial estimate of the state vector $\mathbf{x}_0$ using a Taylor series of first order.

$$\mathbf{g}(\mathbf{x}_{k+1})=\mathbf{g}(\mathbf{x}_k)+\left.\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\right|_{\mathbf{x}_k}.\Delta\mathbf{x}_k \tag{11}$$

denoting

$$\mathbf{g}(\mathbf{x}_{k+1})-\mathbf{g}(\mathbf{x}_k)=\Delta\mathbf{z}_k \tag{12}$$

$$\mathbf{x}_{k+1}-\mathbf{x}_k=\Delta\mathbf{x}_k \tag{13}$$

$$\left.\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\right|_{\mathbf{x}_k}=J(\mathbf{x}_k) \tag{14}$$

the equation that needs to be solved in each iteration is

$$\Delta\mathbf{z}_k=J(\mathbf{x}_k).\Delta\mathbf{x}_k \tag{15}$$

which enables the correction to the state vector to be made

$$\mathbf{x}_{k+1}=\mathbf{x}_k+\Delta\mathbf{x}_k \tag{16}$$

After several iterations the estimate $\hat{\mathbf{x}}$ is reached if $\mathbf{x}_{k+1}$ satisfies equation (10) to within a predetermined accuracy. The convergence property of the technique clearly depends on the sensitivity of the flows to the change in pressures for various network elements. If the Jacobian elements fluctuate excessively with small changes to the state estimate vector, there will be an increase in the number of iterations taken before convergence, due to the iterative corrections

$$\Delta\mathbf{x}_k=J^{-1}(\mathbf{x}_k)[\mathbf{g}(\mathbf{x}_{k+1})-\mathbf{g}(x_k)]$$

being under or over estimated. For highly nonlinear elements, enlisted in equations (1)-(9), which have discontinuous first derivatives, the sensitivity can change quite dramatically. So a smooth approximation to the hydraulic relationships for these network elements is essential for convergence to the solution of the network, as it will discourage oscillation of the state vector values. Three methods of approximation will be discussed in this paper, namely the Fourier series, Lagrange's formula and the Simple Sine approximation.

2. Smoothing Formulae

The model of the water distribution network includes such hydraulic elements as valves and pumps, alongside the more common pipes. This entails approximating the non-smooth curve which describes the relationship between the change in pressure and the flow of water in one of the above elements (see Figure 2). These equations are all non-linear and the majority are discontinuous at certain values of pressure difference. We will use the pressure-reducing valve as an example throughout this paper, see Equation (4).

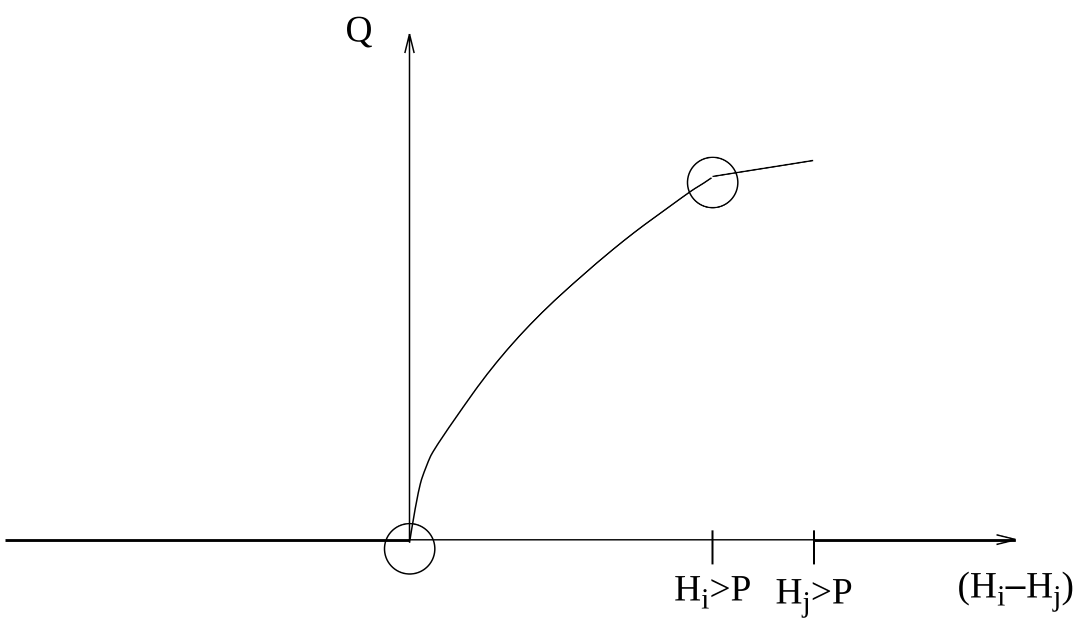
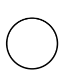


Figure 2. The typical relationship found between the pressure difference and the flow through a pressure reducing valve link

 depicts the discontinuity of the gradient of the curve at these points

Three methods have been considered, namely, the Fourier series, Lagrange's formula and the Simple Sine approximation [2,3]. In all cases, the curve near the discontinuity of the gradient, say at α, is approximated by two straight lines $y=b_1x+c_1$ and $y=b_2x+c_2$, where x and y are equivalent to the difference in pressure $(H_i–H_j)$ and the flow Q, respectively, (see Figure 3a). These lines are evaluated by using the values of the curve at $\alpha–\varepsilon$, α and $\alpha+\varepsilon$ (ε is chosen to coincide with the accuracy of the telemetry measurements). The various formulae are used to approximate (and hence smooth) the above new function between $\alpha–\varepsilon$ and $\alpha+\varepsilon$ (see Figure 3b). The appropriate elements of the Jacobian matrix are replaced with the differential of the formulae used.

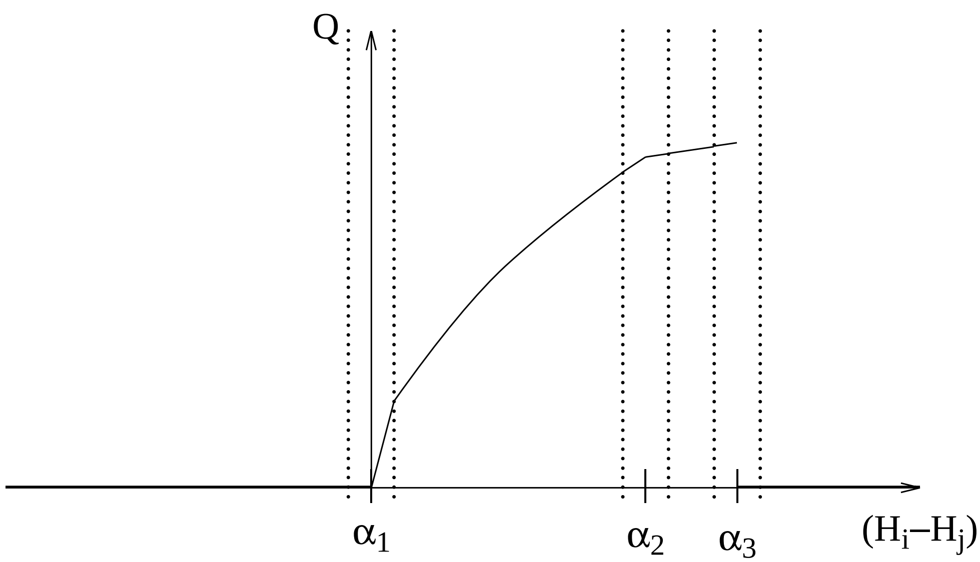


Figure 3a. Approximation of the curve by straight lines, near the vicinity of each irregularity

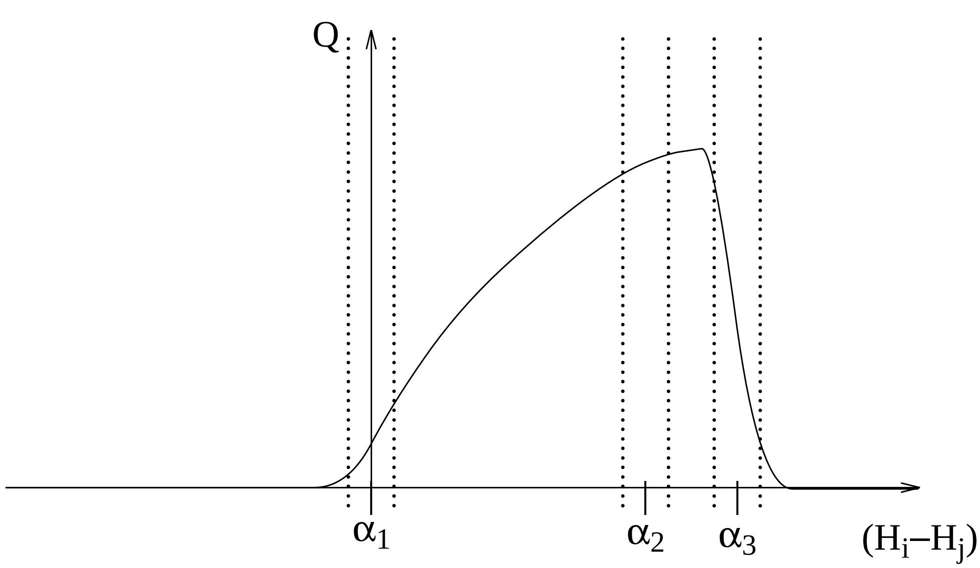


Figure 3b. The smoothed curve

Before the various smoothing formulae can be evaluated, the appropriate order for the resulting polynomial must be determined. Richards [4] suggests a very easy way of deciding upon the order. This is, simply, to evaluate the function at a number of points, and continue to differentiate the function until all of the values are identical:

x	**y**	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$
$\alpha-\varepsilon$	$b_1(\alpha-\varepsilon)+c_1$	b_1	0
$\alpha-\frac{\varepsilon}{2}$	$b_1(\alpha-\frac{\varepsilon}{2})+c_1$	b_1	0
$\alpha^{(-)}$	$b_1(\alpha^{(-)})+c_1$	b_1	0
$\alpha^{(+)}$	$b_2(\alpha^{(+)})+c_2$	b_2	0
$\alpha+\frac{\varepsilon}{2}$	$b_2(\alpha+\frac{\varepsilon}{2})+c_2$	b_2	0
$\alpha+\varepsilon$	$b_2(\alpha+\varepsilon)+c_2$	b_2	0

So the approximating polynomial is chosen to be of order 3.

2.1 The Fourier series

The theory of Fourier series is rather complicated, but the application of these series is simple. Fourier series are powerful tools in treating various problems involving periodic functions. Since, of course, many practical problems do not involve periodic functions, it is desirable to generalize the method of Fourier series to include nonperiodic functions. This is achieved by considering the Fourier series of an arbitrary function of period T and allowing T to approach infinity. This results in the Fourier integral.

$$g(x) = \frac{1}{\pi}\int_0^\infty [\,A(w)\cos wx + B(w)\sin wx\,]\,dw \qquad (17)$$

with

$$A(w) = \int_{-\infty}^{\infty} y(v)\cos wv\,dv \qquad (18)$$

$$B(w) = \int_{-\infty}^{\infty} y(v)\sin wv\,dv \qquad (19)$$

where

$$y(x)=\begin{cases} b_1x+c_1 & \alpha-\varepsilon\le x<\alpha \\ b_2x+c_2 & \alpha<x\le\alpha+\varepsilon \\ 0 & x<\alpha-\varepsilon \\ 0 & x>\alpha+\varepsilon \end{cases}$$

This is the method of Fourier series for nonperiodic functions [3]. At a point where y(x) is discontinuous, the value of the Fourier integral equals the average of the left-hand and right-hand limits of y(x) at that point. It approximates the pressure reducing valve equation as follows:

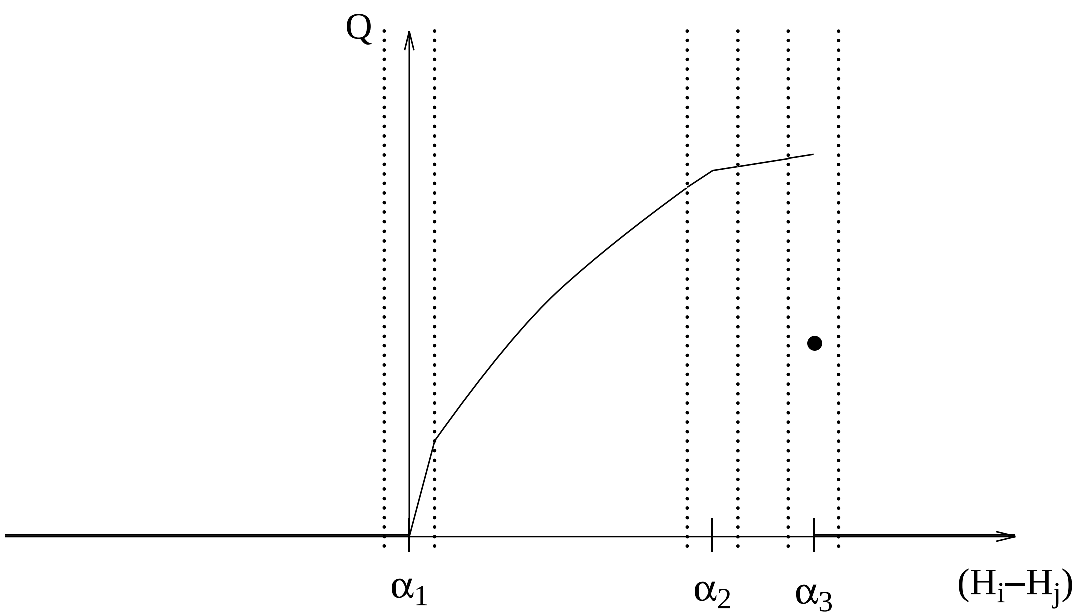


Figure 4a. Approximation of the curve by the Fourier series near the vicinity of each irregularity

2.2 The Lagrange formula

The Lagrange formula is one method of interpolation. The usual methods of interpolation are based on the assumption that the function f(x) can be approximated by a polynomial p(x). The known values of f(x) are called the pivotal values, at these pivotal points f(x)=p(x). The Lagrange formula has the advantage that the calculation of differences - $f(x_{k+1})-f(x_k)$ - which is a normal procedure for interpolation, is not necessary. The Lagrange formula is written as

$$g(x)=\sum_{i=0}^{n} y_i l_i(x) \qquad (20)$$

$$l_i(x) = \prod_{j \neq i} \left(\frac{x - x_j}{x_i - x_j} \right) \qquad i=0,1,...,n \tag{21}$$

The best order for the approximating polynomial is assumed to be 3, the Lagrange formula is then easily calculated using n from 0 to 3. The x_i are issued values which are evenly spaced throughout the interval $[\alpha-\varepsilon, \alpha+\varepsilon]$; the y_i are the corresponding values of flow through the link. The approximation, by the Lagrange formula, of the pressure-flow relationship for the pressure-reducing valve looks like this in graphical form:

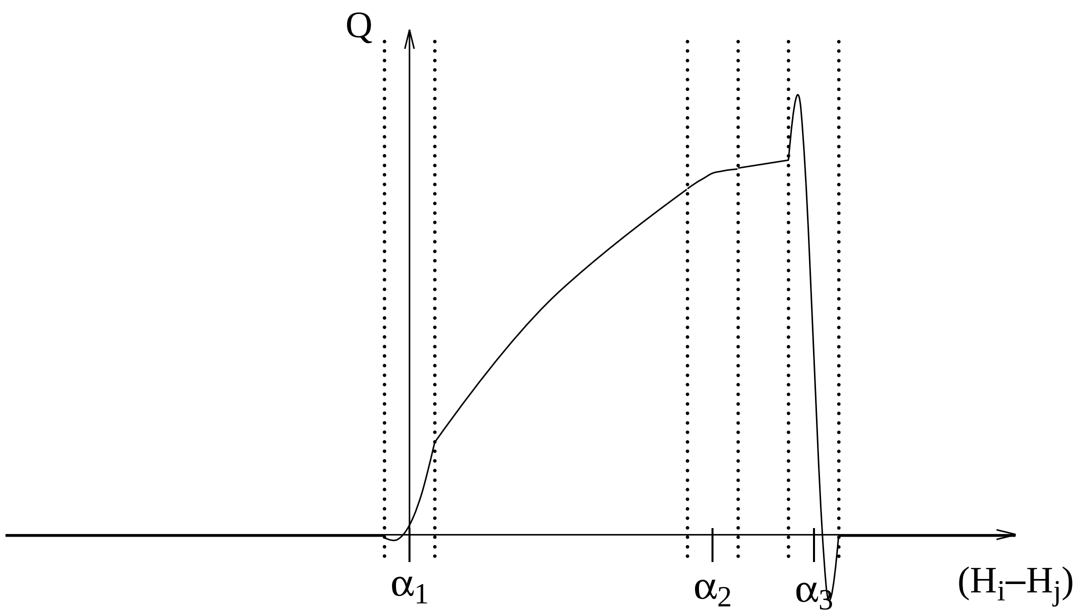


Figure 4b. Approximation of the curve by the Lagrange formula near the vicinity of each irregularity

2.3 The Simple Sine approximation

Then, finally, the Simple Sine approximation is investigated. The β and γ are chosen to ensure that the value and gradient of g(x) are equal to the value and gradient of y(x) at the end-points, $\alpha \pm \varepsilon$.

$$g(x) = a_0 + a_1 \sin\left(\frac{x - \beta}{\gamma} \right) \tag{22}$$

with

$$\begin{aligned} y_{max} &= a_0 + a_1 \\ y_{min} &= a_0 - a_1 \end{aligned} \tag{23}$$

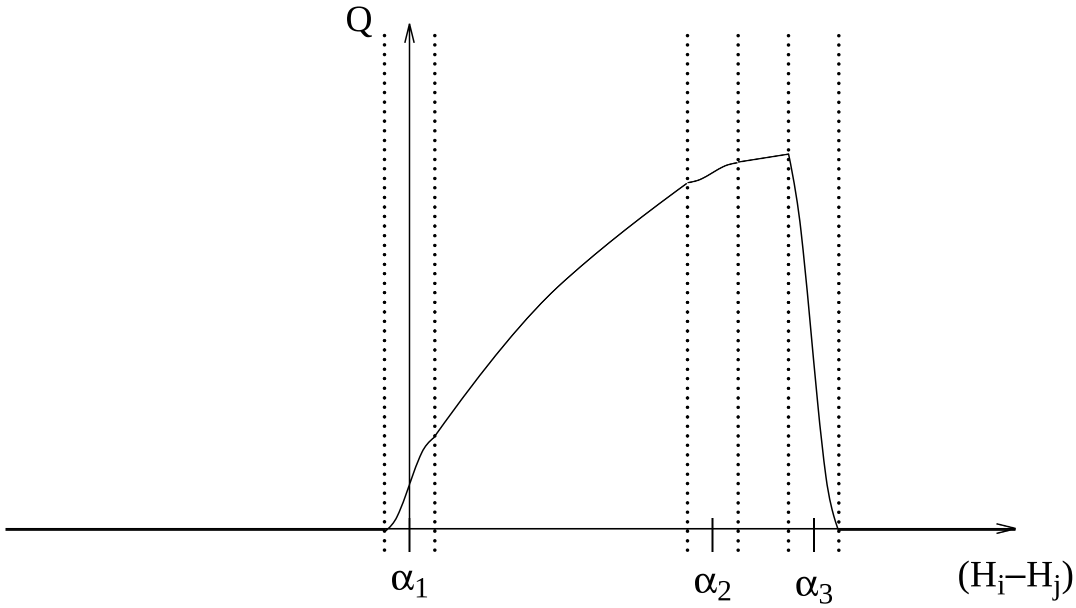


Figure 4c. Approximation of the curve by the Simple Sine approximation near the vicinity of each irregularity

3. Test Results

The effectiveness of each approximation has been tested independently on a small network of 11 nodes, with data similar to those used in [5]. The network consists of 11 pipes, 1 one-way valve, 1 pressure-reducing valve and 3 fixed-head pumps. The one-way valve, pressure-reducing valve and fixed-head pumps are all simultaneously approximated by the smoothing formulae - Fourier series, Lagrange formula and Simple Sine approximation, in turn.

The three smoothing methods are compared alongside the original algorithm which uses the unsmoothed formulae for the hydraulic relationships of the network elements. This inclusion of the original formulation is intended to give an insight into the necessity (or to the contrary) of using the smoothing formulae for solving those networks which are comprised of hydraulic elements such as pumps and valves.

A number of examples have been constructed by modifying the original data set (see Table 1). Example 1 considers the network shown in Table 1, an 11-node network with one fixed-head node (at node 2). Each of the smoothing formulae take the same number of iterations to converge as the original formulation. For Example 2 the value of the pressure limit for the pressure-reducing valve is decreased to 91.16m. This reduction of the pressure limit ensures that the approximating formulae are going to be heavily in use whilst the network is being solved. The result is that the network with ‘smoothed’ nonlinear elements converges in fewer iterations than the original formulation - so an improvement has been realised. For Example 3 the head difference in the fixed-head pump between nodes 11 and 10 is reduced. The Lagrange formula shows a clear advantage over the other methods; the Simple Sine approximation again takes less iterations than the original formulation, whilst the Fourier series converges in the same number of iterations as the original formulation. In Example 4, the modifications made for Example 2 and Example 3 are combined to illustrate how the different methods perform under more extreme situations. Once again the Lagrange formula converges in the least number of iterations, but this time the Simple Sine approximation takes the same number of iterations to converge as the original formulation, whilst the Fourier series takes more iterations. Then in Example 5, the initial state vector is altered, giving nodes 5 and 6 the same head value, this tests the convergence capability of the one-way valve. The Lagrange formula and the Simple Sine approximation take the same number of iterations to converge while the Fourier series takes several more iterations, but the original formulation does not converge at all - so a marked improvement is evident.

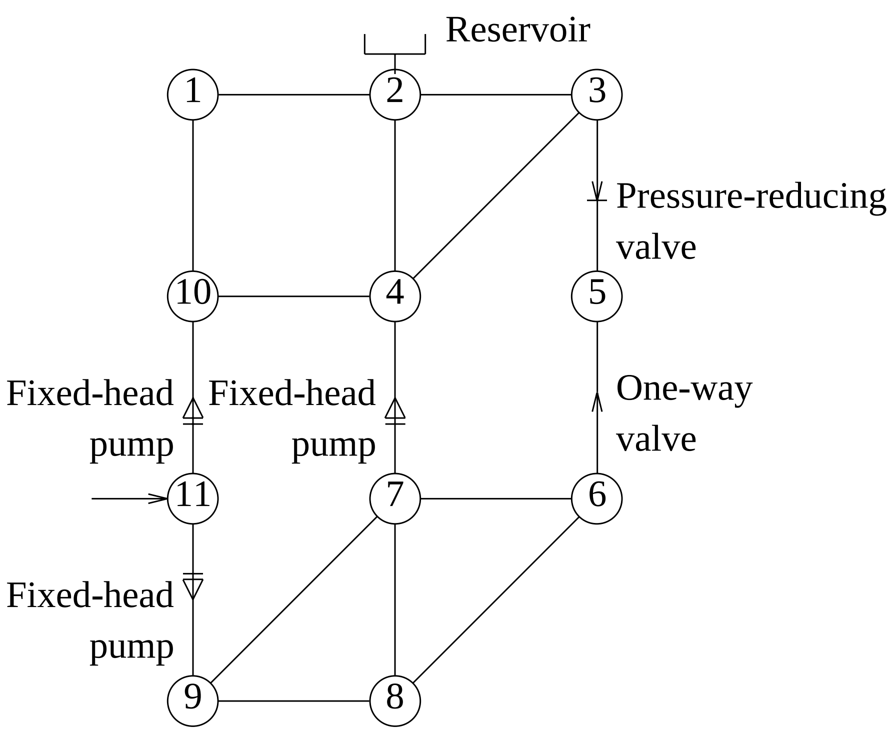


Figure 5. The 11-node network.

TABLE 1. THE ORIGINAL DATA SET
FOR THE 11-NODE NETWORK.

Nodes	Exact state	Measurements/Pseudomeasurements		
		Head, mAq	Load, m^3/s	Inflow, m^3/s
1	145.89		-0.02832	
2	145.76	145.8	-0.02832	
3	144.16		-0.02832	
4	147.01		-0.02832	
5	92.09		-0.05664	
6	93.23		-0.02832	
7	90.56		-0.02832	
8	100.40		-0.05664	
9	115.78		0.0 P	
10	146.31		0.0 P	
11	60.96			0.2832
11	0.2832			

Links	Link parameters			
Pipe N_1–N_2	Length, m	Diameter, m	CHW	
10 - 1	914.4	0.4064	100	
2 - 1	914.4	0.3048	120	
3 - 2	609.6	0.2540	110	
4 - 2	609.6	0.3048	115	
10 - 4	609.6	0.3048	110	
4 - 3	609.6	0.2540	100	
6 - 7	609.6	0.2540	110	
8 - 7	609.6	0.2032	100	
9 - 8	609.6	0.3048	110	
7 - 9	1219.2	0.2540	100	
8 - 6	609.6	0.2540	120	
Fixed-hd. pump	Length, m	Diameter, m	CHW	Head difference, mAq
7 - 4	609.6	0.2540	289	60.96
11 - 10	609.6	0.2540	117	90.39
11 - 9	304.8	0.4064	65	65.52
One-way valve	Length, m	Diameter, m	CHW	
6 - 5	1219.2	0.2032	100	
Pressure valve	Length, m	Diameter, m	CHW	Pressure limit, m
3 - 5	1219.2	0.2032	110	109.72

P - Pseudomeasurement

CHW - Hazen-Williams coefficient

N_1–N_2 - sending and receiving nodes of the pipe

1 mAq=$9.81\times10^3 Nm^{-2}$ - head units

TABLE 2. THE NUMBER OF ITERATIONS REQUIRED FOR CONVERGENCE OF THE DIFFERENT FORMULATIONS.

	Number of iterations, using approximations by:			
Example number	Fourier	Lagrange	Sine	Original
1	2	2	2	2
2	8	8	8	12
3	16	6	10	16
4	16	8	12	12
5	16	12	12	>100

4. Conclusion

Several methods of approximating the nonlinear equations, which describe the relationship between pressure difference and flow for links in the network, have been discussed. The aim was to ensure a more rapid convergence of the solution of a network consisting of a number of hydraulic elements. Such a method has been found and tested. The Lagrange formula has been shown to be the most suitable approximation for the hydraulic relationships at points of discontinuity and general ‘unsmoothness’. The results obtained indicate the convergence rate and the robustness of the algorithm.

5. References

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