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# Fuzzy fractal dimensions and fuzzy modeling

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**Abstract**

Fractal dimensions describe self-similarity (structural complexity) of various phenomena (such as e.g., temporal signals, images). Their determination (through box counting or a correlation method) is inherently associated with the use of information granules—sets. The intent of this study is to generalize the idea to the domain of fuzzy sets and reveal associations between the mechanisms of fractal analysis and granular computing (including fuzzy modeling). First, we introduce the concept itself and discuss the role of fuzzy sets as a vehicle for constructing fractal dimensions. Second, we propose an algorithmic framework necessary to carry out all computing. and experimentally quantify a performance of regression models used to determine fractal dimensions and contrast it with the performance of the fractal models existing in the set-based environment. It is shown that fuzzy set approach produces more consistent models (in terms of their performance). We also postulate a power law of granularity and discuss its direct implications in the form of a variable granularity in fuzzy modeling. In particular, we show how the power law of granularity helps construct mappings between system's variables in rule-based models. Experimental studies involving several frequently used categories of fuzzy sets illustrate the main features of the approach.


*Keywords:* Fractals; Self-similarity; Structural complexity; Fractal dimension; Generalized correlation fractal dimension; Power law of information granularity; Variable granularity in fuzzy modeling

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## 1. Introduction

Fractal analysis has been found useful in description of dynamic phenomena [1,4,7,8]. It has also resulted in a number of applications, cf. [2,4–6]. The term itself pertains to the characterization of a global property of self-similarity that describes how much a part of a phenomenon can be scaled up to a whole. An instrument used to quantify this effect is a fractal dimension [4]. In time series, for instance, a fractal dimension expresses the regularity of the series and states how similarity scales up when such time series is observed over a longer time interval. The self-similarity could be also regarded as a measure of geometrical complexity of an object under discussion. The determination of the fractal dimension is inherently associated with set-based constructs. The generic box dimension [7] measures in which way the number of occupied boxes (viz. those including the elements of the time series) increases when the size of the box decreases. The other common techniques of fractal determination uses a so-called correlation dimension in which a count of elements concerns a family of spheres (hyperspheres) constructed around each data point. What is common to the existing techniques (in spite of evident technical differences) is that all of them exploit *sets* regarded as information granules that allow us to see only a certain part of the phenomenon. The changes in the size of the information granules imply how large part we are taking into consideration.

Granular computing is a cornerstone of processing endeavors in intelligent systems [9]. Human perceive the word, organize knowledge and make it highly operational by forming and manipulating information granules. Information granulation is an example of abstraction. There are numerous facets of the granular information processing as well as there is a variety of formal frameworks in which such information granulation takes place. For instance, these include set theory, fuzzy sets, random sets, rough sets and many others.

It becomes quite obvious that there could exist some interesting and potentially useful links between the fundamental concepts of fractals and granular computing. Fractals use information granules to determine their fundamental characteristics. Reciprocally, we can exploit ideas of fractals in the development of granular models [3].

In this study, our objective is to reveal links between fractal analysis and the role of information granules that is played there. In particular, we revisit an underlying idea of computing a fractal dimension involving fuzzy sets. An interesting question pertains to the relationship between the fractal dimension and the form of fuzzy sets used as focal elements in the calculations of the fractal dimension. The other one closely related pertains to the quality of the assumed relationship governing the form of self-similarity in systems. The intent is to raise awareness of this issue, focus on the underlying generalization and come up with some algorithmic details.

In the paper, we confine ourselves to time series. This is primarily dictated by the existing applications in which time series arise naturally as well as the clarity of the fractal concept itself that manifests profoundly in this setting. Nevertheless the proposed methodology readily expands to other higher dimensional objects (such as images or multivariable systems).

The material is arranged into five sections. First, in Section 2 we discuss how fuzzy sets augment the concept of the fractal dimension and contribute to the computing side. The intent of Section 3 is to identify a number of open issues related to this granular framework. Numerical experiments are covered in Section 4.

## 2. Fractal computing with fuzzy sets

In this section, we review the main constructs of fractal analysis and then proceed with their generalization in terms of information granules expressed in the language of fuzzy sets.

Consider a time series $\{t(k), x(k)\}$, $k = 1, 2, \ldots, N$, where $x(k) \in \mathbf{R}$ and $t(1), t(2), \ldots, t(N)$ denote discrete time moments in which the values of this time series are recorded. In this sense, we are provided with a collection of two-dimensional elements $\mathbf{S} = \{(t(k), x(k))\}$, $k = 1, 2, \ldots, N$. The structural complexity of $\mathbf{S}$ is measured by a fractal dimension $D^\wedge$ defined as the following limit

$$D^\wedge = \lim_{\varepsilon \to 0} \frac{\log(N(\varepsilon))}{\log(\varepsilon)} \tag{1}$$

where $N(\varepsilon)$ is a number of boxes of size $\varepsilon$ used to cover the object (here the given time series). In essence, the above relationship relates to the power law stating that $N = \varepsilon^{D^\wedge}$. In practice, the fractal dimension has to be estimated with the use of some experimental data. A collection of "c" experiments concerns a determination of the number of boxes $N(\varepsilon)$ for a given value of the size of the box, see Fig. 1. Then these experimental pairs $(\varepsilon_j, N(\varepsilon_j))$, $j = 1, 2, \ldots, c$ are used to determine parameters of the linear model. It can be easily shown that in a double logarithmic model of the form

$$\log \widehat{y} = D \log(\varepsilon) + C$$

the fractal dimension ($D$) appears as a slope of the computed regression line. The regression model itself is constructed through a minimization of the well-known performance index $Q$ treated as a sum of squared errors

$$Q = \sum_{k=1}^{c} (\log N(\varepsilon_k) - D \log(\varepsilon_k) - C)^2 \tag{2}$$

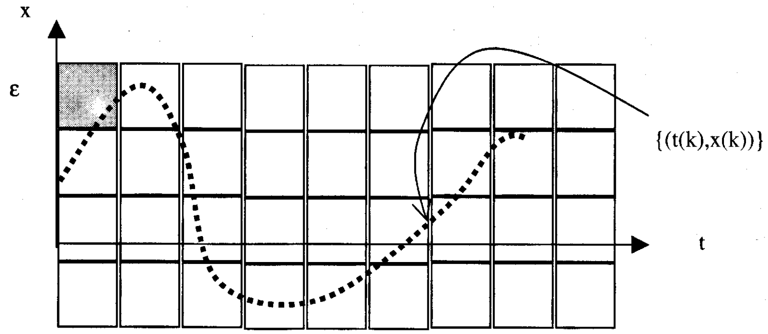


Fig. 1. An example of box counting leading to the determination of the fractal dimension.

Conceptually, the box method being illustrated in Fig. 1 is the most intuitive approach to the determination of the fractal dimension. Unfortunately, computational problems arise when dealing with higher dimension of the objects that are situated in $\mathbf{R}^n$. In this case one usually proceeds with a correlation dimension [4] where we construct hyperspheres around the individual points of the data set, see Fig. 2. Following this way, the time complexity depends on the number of data elements. Following this method, we count the number of data points residing within a sphere of radius $\varepsilon$. Then the total number of points covered by the spheres $N(\varepsilon)$ is equal to

$$N(\varepsilon) = \frac{1}{N(N-1)} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \Omega_{ij}(\varepsilon) \tag{3}$$

where $\Omega_{ij}(\varepsilon)$ is a sphere (or a hypershpere in a multidimensional case) defined as follows

$$\Omega_{ij}(\varepsilon) = \begin{cases} 1 & \text{if } \sqrt{(t(i)-t(j))^2 + (x(i)-x(j))^2} \leqslant \varepsilon \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

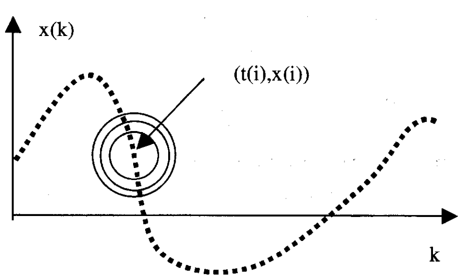


Fig. 2. Computing correlation fractal dimension; $\Omega_{ij}(\varepsilon)$ denotes a sphere constructed around data point $(t(i), x(i))$.

Next, the detailed computations of the fractal dimension are carried out on a basis of the regression model formed for the resulting pairs $(\varepsilon, N(\varepsilon))$.

What needs to be emphasized is a fact that the estimation process dwells directly on a sequence of information *granules* that are used directly to determine the fractal dimension. The methods exploited so far are confined primarily to sets regarded as basic granular instruments being applied to quantify the structural properties of the phenomenon (signal) come in the form of sets (in spite of the detailed geometry that is boxes or hyperspheres). The generalization one may think of is straightforward: rather than confining to sets, we consider any other information granules using which we look into the matter of scaling. In particular, such information granules can be treated as fuzzy sets. This becomes even more legitimate as we are concerned with these granules as a description and modeling language for such numeric data. The definitions introduced above require some reformulation yet this is quite straightforward. For instance, in the case of the computing of the correlation dimension now illustrated in Fig. 3, we find that a fuzzy cardinality ($\sigma$-count) of a fuzzy set measuring the scaling property becomes involved in the computations of the fractal dimension. More specifically, we have

$$N(\varepsilon) = \frac{1}{N(N-1)} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \Omega_{ij}(\varepsilon) A_{ij}(\varepsilon) \tag{5}$$

It is instructive to elaborate more on the function of the fuzzy set $A_{ij}$ as it shows up in the above counting. It is defined over the space of amplitudes of the signal and centered at $x(i)$. Its primary role is to distinguish between data elements contributing to the overall count (5).

As an example, consider a Gaussian membership function

$$A_{ij}(\varepsilon) = \exp(-(x(i) - x(j))^2/\varepsilon^2)$$

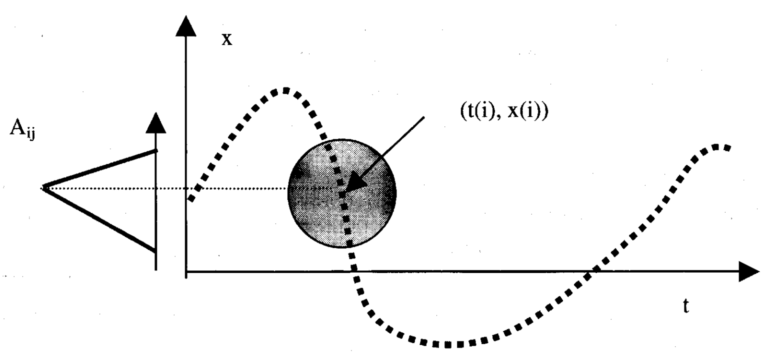


Fig. 3. Computing fractal dimension with the use of the correlation dimension in the framework of fuzzy sets used as information granules.

Assuming $x(i)$ to be fixed, the membership function returns a degree of membership of $x(j)$ to this neighborhood. In this sense, it helps distinguish (discriminate) between various elements of the time series falling into the given window $\Omega_{ij}(\varepsilon)$. The larger the membership grade of $A_{ij}$ for the given $x(j)$, the more visible its contribution to the overall sum $N(\varepsilon)$. The membership function itself is equipped with a spread parameter ($\varepsilon$) that controls the size of the information granule. One should stress that the fuzzy sets in the above definition (5) are defined in the amplitude space while the time variable over which all discrete time points have been distributed is not affected and the granulation there is related to the window (we have $\Omega_{ij}$ in place and this confines the overall collection of experimental data taken into consideration).

We can look at (5) in a different way by writing it down as follows

$$N(\varepsilon) = \frac{1}{N(N-1)} \sum_{i=1}^{N} \sigma(A_i, \varepsilon) \tag{6}$$

Here $\sigma(A_i)$ can be treated as a modified (truncated) $\sigma$-count of $A_{ij}$ where the truncation is realized in the format

$$\sigma(A_i, \varepsilon) = \sum_{\substack{j=1 \\ j \neq i}}^{N} \Omega_{ij}(\varepsilon) A_{ij}(\varepsilon) \tag{7}$$

In other words, $\sigma(A_i, \varepsilon)$ summarizes the membership grades of the elements ($x(j)$) that fall within the "scope" of $\Omega_{ij}(\varepsilon)$.

## 3. The research issues dealing with the fractal dimension

In addition to the increased conceptual coherency (the usage of the same formal apparatus of information granulation across all modeling phases), there are a number of issues worth emphasizing. Some of these can be quantified primarily on an experimental basis (as we may not have enough fundamental underpinnings available). Some others could lead to the better understanding of the role of fuzzy sets in information granulation and granular modeling.

(a) *diversity of information granules* (classes of membership functions). Fuzzy sets offer a variety of forms of information granules by choosing a certain family of membership functions. In other words, the value of the fractal dimension becomes associated with the given category of fuzzy sets (we may talk about the quantization of the structural and self-similar aspects of the data in this particular context).

(b) performance of the approximation model of the fractal dimension. As stated above, the value of the fractal dimension is obtained through finding a regression line representing the experimental data. The performance of the model ($Q$) depends on the class of the membership functions being used in the determination of the fractal dimension.

(c) linkage between the fractal dimension and the number of information granules required to use in modeling activities. Intuitively, the more complex the phenomenon to be modeled, the more information granules one has to use to construct a model. The conjuncture we make is that the fractal dimension determines the number of the fuzzy sets and the relationship is a power law. More precisely, if we admit a certain maximal number of fuzzy sets to be used in the modeling process that is equal to $N_{\text{nominal}}$ then the number of fuzzy sets to be used $N$ is equal to

$$N = N_{\text{nominal}}^{D}$$

As an example, let us set $N_{\text{nominal}}$ to be equal to 5. Then the values of $N$ for selected values of $D$ are collected below

| $D$ | 0.40 | 0.90 | 1.20 | 1.60 | 1.90 | 2.00 |
|---|---|---|---|---|---|---|
| Number of fuzzy sets (rounded up to the larger integer) | 2 | 5 | 7 | 13 | 21 | 25 |

(d) Expressing variable complexity of the time series in modeling activities. This may lead to a different number of information granules to be used in the fuzzy model. The fractal dimension can be computed for each fixed point of the time series and used as an index of "local" complexity that converts into the corresponding size of the vocabulary of information granules. Following the power law, the higher the local fractal dimension, the more fuzzy sets need to be allocated to the local model constructed. Lower values of the fractal dimension require less fuzzy sets to be introduced for modeling purposes.

## 4. Experimental studies

Two experiments that are based on the data available on the WWW help quantify the conjectures made in the previous section. The third one uses synthetic data implied by some one-argument function. In all experiments we discuss three classes of fuzzy sets (membership functions) (Table 1).

All the data sets are normalized to the unit interval.

*Synthetic data* (http://lib.stat.cmu.edu/datasets/sapa). This is a simulated AR(2) time series consisting of 1024 samples, Fig. 4.

The collections of results $\{\log(\varepsilon), \log(N(\varepsilon))\}$ are organized in Fig. 5. In general, they follow a straight line yet the approximation error varies and

Table 1
Classes of fuzzy sets ($A$); $\varepsilon$ pertains to the size of the information granules while "$m$" localizes them in the universe of discourse

| Class of fuzzy sets | Description |
|---|---|
| Triangular | $A(x) = \begin{cases} 1 - \frac{\lvert x-m \rvert}{\varepsilon} & \text{if } \lvert x-m \rvert \leqslant \varepsilon \\ 0 & \text{otherwise} \end{cases}$ |
| Parabolic | $A(x) = \begin{cases} 1 - \frac{(x-m)^2}{\varepsilon^2} & \text{if } \lvert x-m \rvert \leqslant \varepsilon \\ 0 & \text{otherwise} \end{cases}$ |
| Gaussian | $A(x) = \exp(-\frac{(x-m)^2}{\varepsilon^2})$ |

depends what type of information granules have been used. Table 2 provides a detailed quantification of the approximation as well as provides with the determined fractal dimension. The fractal dimensions do not vary substantially from method to method yet the approximation is better when using fuzzy sets. Interestingly, the use of the triangular fuzzy sets yields the same results as sets.

*Fraser River data.* This set of data (time series) represents mean monthly flow of Fraser River at Hope, B.C., expressed in cms, March 1913–December 1990. The normalized data are visualized in Fig. 6.

The calculations of the fractal dimension are carried out in the same way as before that is using fuzzy sets with triangular, parabolic and Gaussian membership functions. The plots of the values of $\log(N(\varepsilon))$ versus $\log(\varepsilon)$ are shown in Fig. 7. The results are summarized in the tabular form by including the

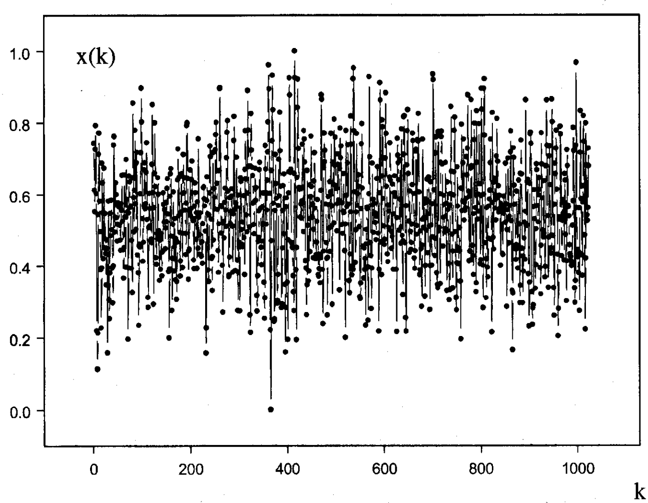


Fig. 4. AR(2) synthetic data.

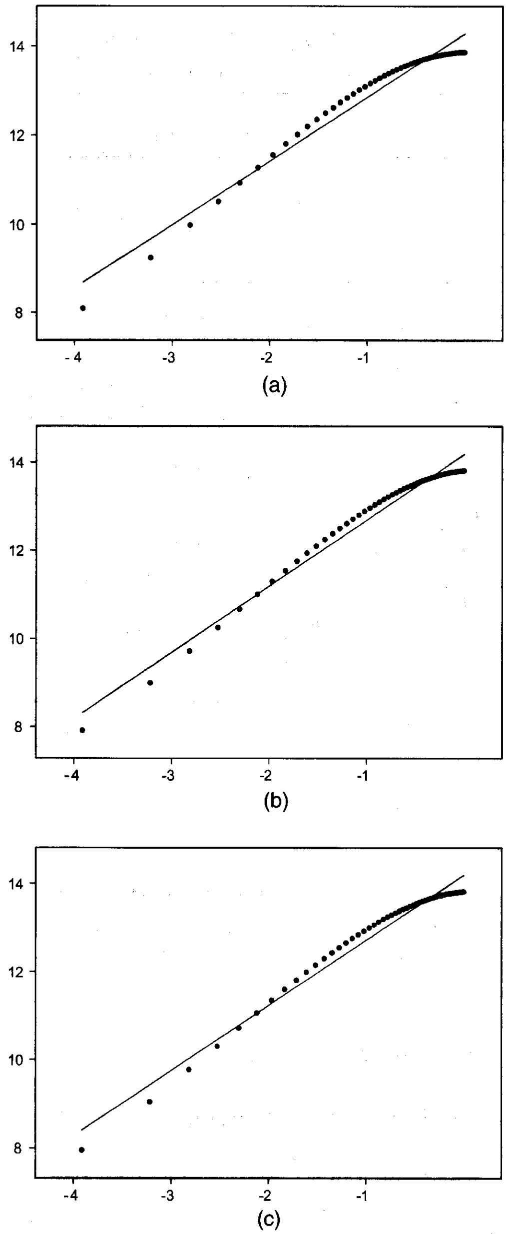


Fig. 5. Plots of $\log(\varepsilon)-\log(N(\varepsilon))$ for various classes of information granules sets (a) parabolic fuzzy sets (b) and Gaussian fuzzy sets (c).

Table 2
Computing of the fractal dimension with the use of set and fuzzy set information granules

| Information granules | Sets | Parabolic fuzzy sets | Gaussian fuzzy sets |
|---|---|---|---|
| Fractal dimension | 1.4287 | 1.5002 | 1.4803 |
| Performance index $Q$ | 0.2399 | 0.2009 | 0.2096 |

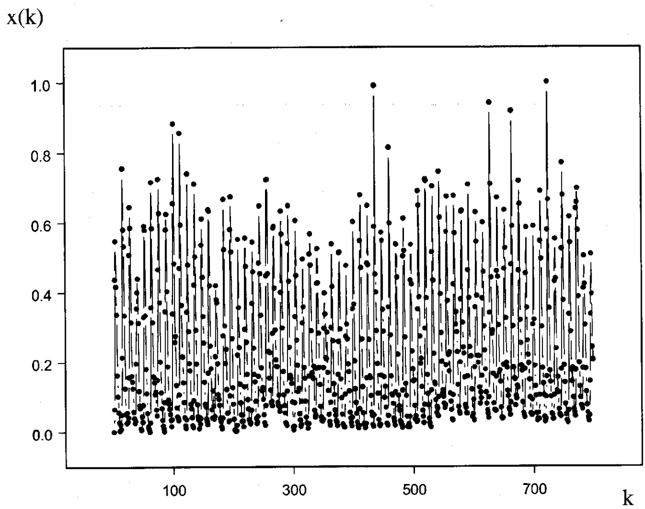


Fig. 6. Normalized time series.

values of the fractal dimension along with the resulting performance index. Both types of fuzzy sets used for fractal determination do a better job than sets. As in the previous example the triangular fuzzy sets yield identical results as sets so in this construct they behave in an analogous manner (Table 3).

Fig. 8 illustrates the values of the local fractal dimensions. While exhibiting a high level of variability, this plot identifies several points with high values of the fractal dimension pointing at the regions of the time series with the highest value of geometrical complexity.

To gain a better insight into the distribution of the values of the local fractal dimension, a plot of the probability density function is helpful (Fig. 9).

*Synthetic data* come from the source governed by the equation

$$x(k) = \mathrm{abs}(\sin(k * \omega)$$

In the computations of the fractal dimension we use the parabolic fuzzy sets. The parameter ($\omega$) is set to 0.005. 800 data points used in the experiment ($N = 800$) (Fig. 10).

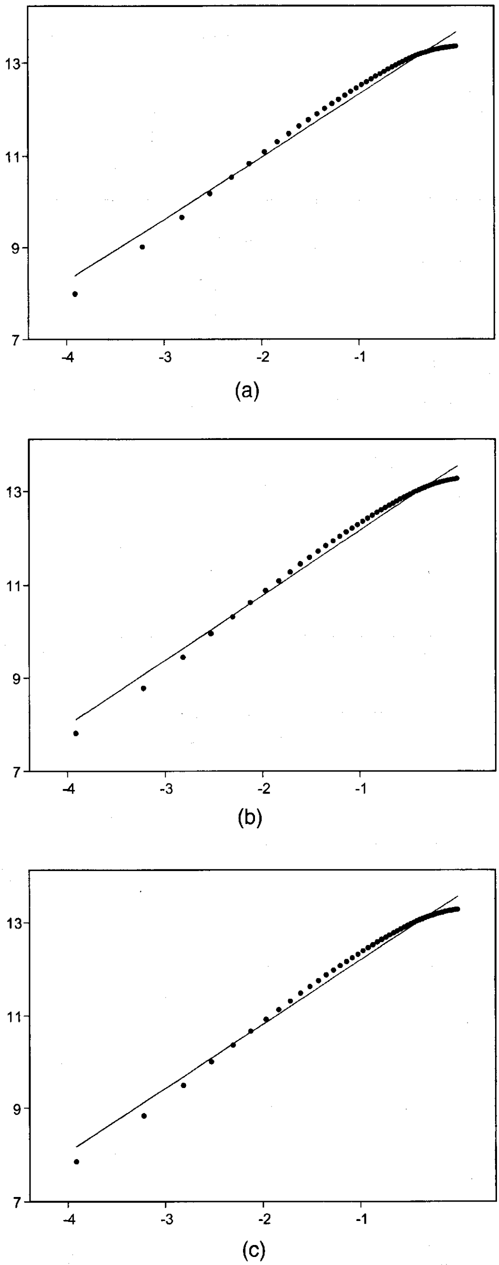


Fig. 7. Plots of $\log(N(\varepsilon))$ versus $\log(\varepsilon)$ for various types of granular constructs (a) sets (b) parabolic fuzzy sets (c) Gaussian fuzzy sets.

Table 3
Results of fractal dimension determination with the use of set and fuzzy set information granules

| Information granules | Sets | Parabolic fuzzy sets | Gaussian fuzzy sets |
|---|---|---|---|
| Fractal dimension | 1.3546 | 1.3897 | 1.3800 |
| Performance index $Q$ | 0.1623 | 0.1428 | 0.1482 |

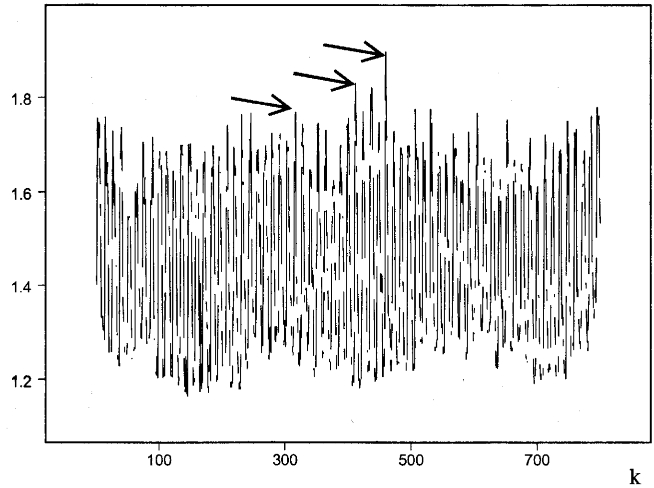


Fig. 8. Local fractal dimension and its distribution over the time series (parabolic fuzzy sets). Several points with the highest local fractal dimension have been identified.

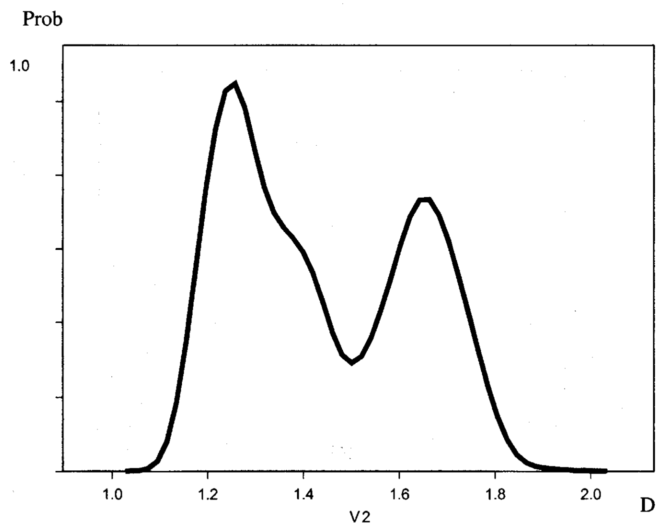


Fig. 9. Probability density function of the local fractal dimension (parabolic fuzzy sets).

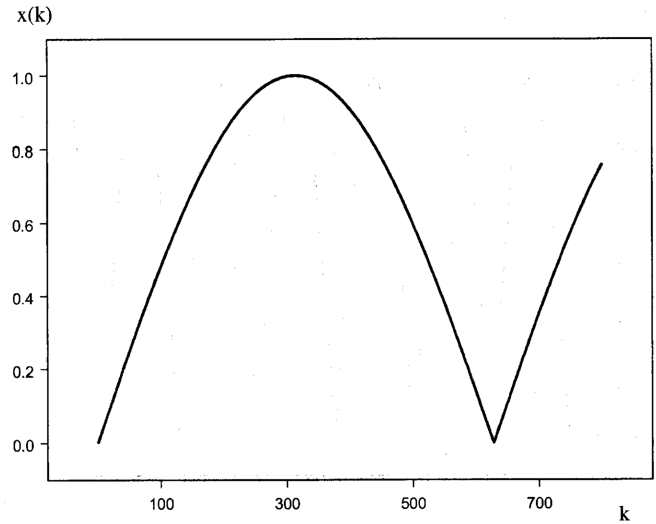


Fig. 10. Synthetic time series $x(k)$.

The value of the fractal dimension $D$ is equal to 1.075833. Next we change the value of $\omega$ by increasing it four times (Fig. 11). The fractal dimension nicely reflects the increased complexity (self-similarity) of the signal producing the value of $D$ equal to 1.397913.

Finally, we experimented with $\omega = 0.050$ (Fig. 12) with the fractal dimension equal now to 1.453097.

The dependence between $\omega$ and the resulting fractal dimension are collected in Table 4. The same table includes the number of the fuzzy sets ($C$) required to model the system assuming that the nominal value $N_{\mathrm{nom}}$ has been set to 4.

Now we consider a mixture of two signals of two different frequencies ($\omega = 0.05$ and 0.005) (see Fig. 13).

The resulting fractal dimension is equal to 1.103449. The local fractal dimension distributed along the time axis reflect the structural complexity (self-similarity) of the signal (Fig. 14). The range of the values of this dimension as well as the associated values of the number of the fuzzy sets are collected in Table 5. The distribution of these values is represented in a histogram form (Fig. 15).

This experiment sheds light on an important role of the fractal analysis of data. Before proceeding with a detailed estimation of the parameters of any fuzzy (granular) model, it becomes obvious that one should invest into understanding the character of the data and allocate various level of information granules into particular regions of data (Fig. 16).

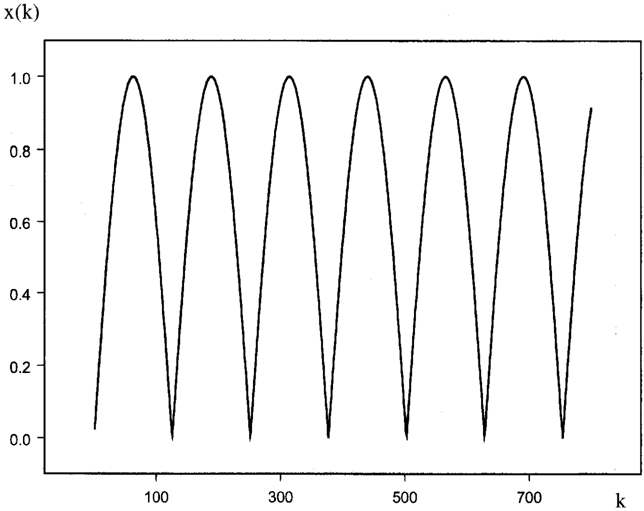


Fig. 11. Synthetic data with $\omega = 0.025$.

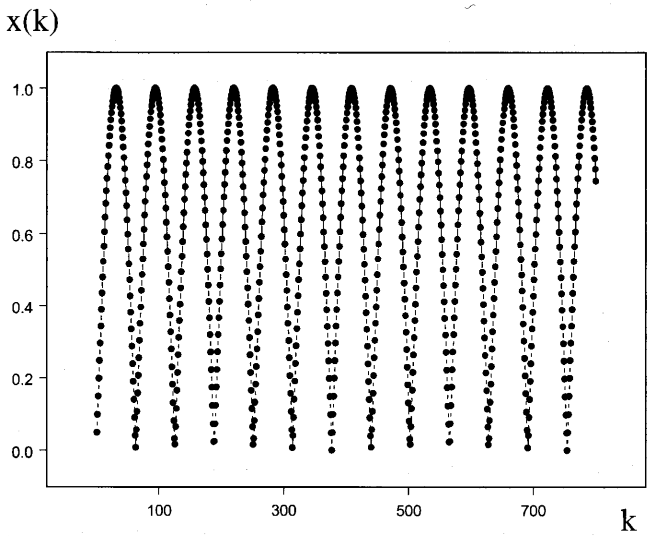


Fig. 12. Synthetic data with $\omega = 0.050$.

Table 4
$D$ and $C$ for selected values of $\omega$

| $\omega$ | 0.005 | 0.025 | 0.050 |
|---|---|---|---|
| $D$ | 1.075833 | 1.397913 | 1.453097 |
| $C$ | 5 | 7 | 8 |

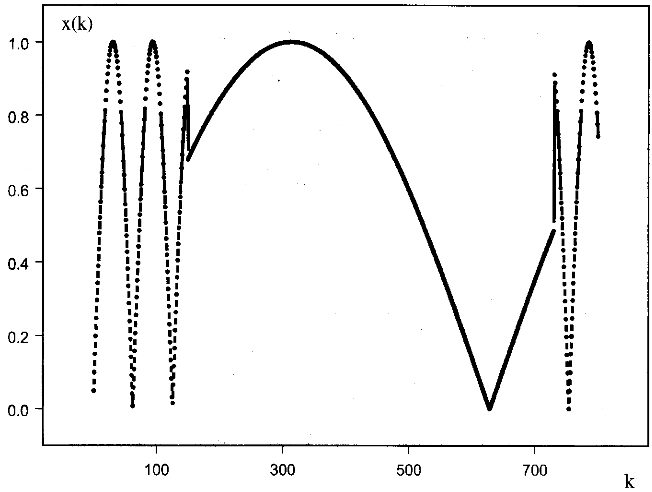


Fig. 13. Synthetic data being a mixture of signals with two different values of $\omega$.

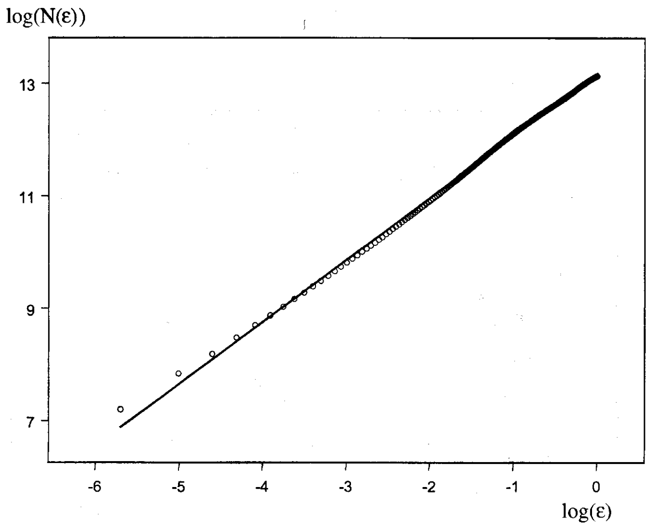


Fig. 14. Plot of experimental data points $(\log(\varepsilon), \log(N(\varepsilon)))$ along with their linear approximation.

Consider a situation where we are interested in the rule-based style of model of a single input–single output relationship. The computations of the local fractal dimension helps us identify the regions in the data space where we need more "detailed" information granules and consequently the number of the rules required to model the data over there becomes higher. The power law of

Table 5
Values of the local fractal dimension (minimal, maximal, first and third quartile) along with the corresponding number of the linguistic labels (for $N_{nom} = 5$)

| $D$ (min) | $D$ (first quartile) | $D$ (mean) | $D$ (third quartile) | $D$ (Max) |
|---|---|---|---|---|
| 0.7488800 | 0.9986852 | 1.2029516 | 1.3468990 | 1.8290410 |
| $N = 4$ | $N = 5$ | $N = 7$ | $N = 9$ | $N = 19$ |

$N$ is rounded up to the larger integer value.

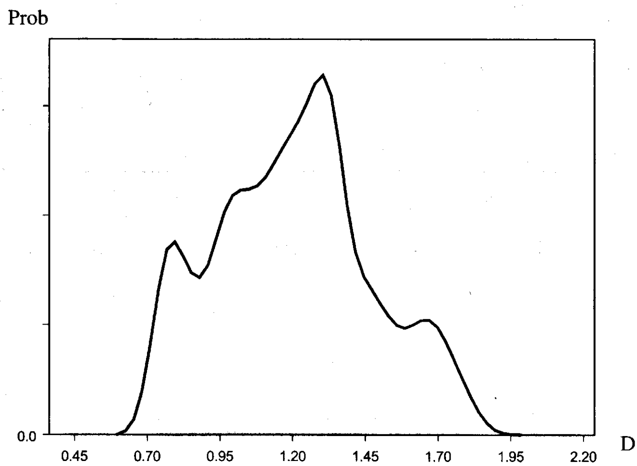


Fig. 15. Histogram of the local fractal dimension.

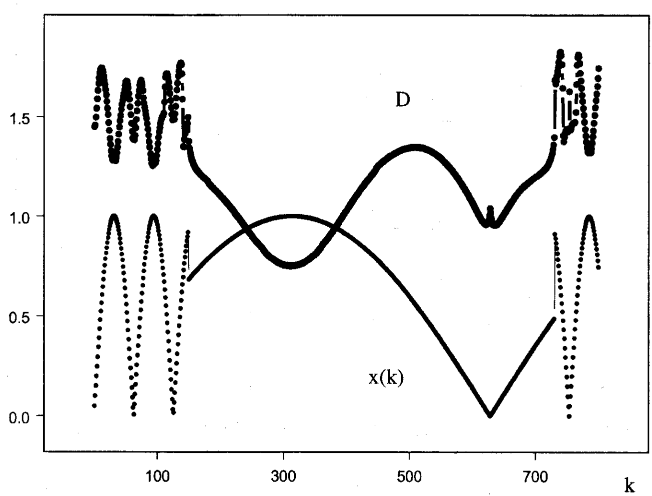


Fig. 16. Values of the local fractal dimension vis-à-vis the original data.

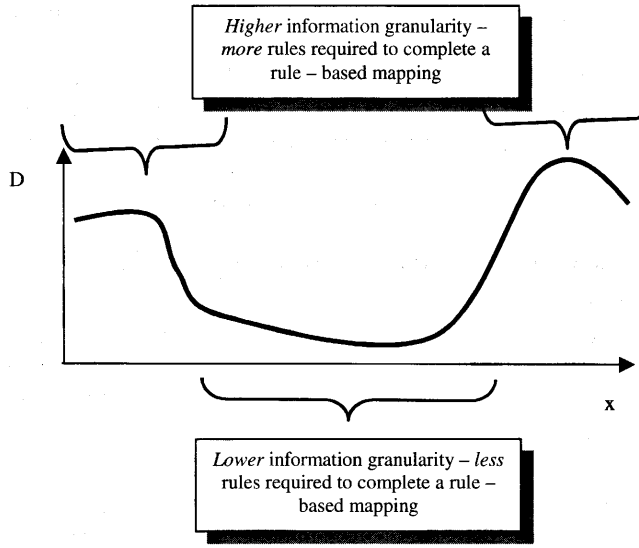


Fig. 17. Single input–single output mapping realized in the form of a rule-based system along with a varying size the information granules across the data space where the allocation is guided by the values of the local fractal dimension.

information granularity becomes useful in allocating the number of fuzzy sets across the data regions (see Fig. 17). One should stress however that this important guideline does not address the details as to the distribution of the fuzzy sets within the given region. This is postponed to the next more parameter-specific phase of system identification.

## 5. Concluding comments

In this study, we have introduced a concept of the fuzzy fractal dimension and showed its role in analysis of granular properties of data as well as proposed a power law of information granularity in the setting of fuzzy modeling. The fractal dimension itself has been linked with the classes of fuzzy sets (membership functions). We have shown that the use of fuzzy sets leads to more consistent regression models using which the fractal dimension is computed. There is an association between the type (membership function) of the information granules being used in the underlying granulation of data and the value of the fractal dimension itself. Furthermore it is shown that triangular fuzzy sets (membership functions) "coincide" with sets when being used in the determination of the fractal dimension. In other words, they are indistinguishable from sets as far as the analysis of the fractal properties of data is concerned.

While the fractal dimensions have already been used in fuzzy system modeling cf. [3] where they are considered as one of the conditions in the rule-based model of a system, we postulate that they can be exploited as a fundamental instrument of granular computing. Especially, we can investigate and reveal the fractal (granular) nature of experimental data before proceeding with detailed fuzzy modeling. We have shown that a general design scheme embodied on the principle of variable granularity can be portrayed in the form of the following scheme.

Data $\rightarrow$ Fractal characteristics of data $\rightarrow$ Variable size of information granules in fuzzy modeling.

The ensuing detailed design of the fuzzy systems exploiting the concept of variable granularity forms a separate subject to investigate and experiment with.

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