# DISCRETE–TIME MULTIVARIABLE SYSTEM DESIGN BY MEANS OF CONTINUOUS–TIME METHODS [1]

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**Abstract.** First, the continuous–time (CT) approximate models of discrete–time (DT) system composed of sampler, zero order hold and CT plant in series are reminded. Then, basing on these models a CT design method of DT controller is described. Next, the same approach is proposed for multivariable system with DT controllers which can have different and asynchronous sampling periods. By the way, the design method of CT multivariable systems basing on design of successive controllers for some replacement SISO plants is proposed. Finally, the proposed design methods are illustrated on an example of twovariable, tworate system.

**Keywords:** Multivariable systems; discrete-time systems; control system design; microprocessor control; multirate systems.

## 1. INTRODUCTION

Lastly in industrial applications the controllers are very frequently realized with using microprocessors. At the same time, the relatively high sampling frequencies are applied in order to improve the quality of control. It is known that such control systems have the properties similar to that of continuous-time (CT) systems. However the controllers realized on microprocessors are described by some discrete-time (DT) algorithms which must be taken into account in the process of design. In the case of multivariable (MV) systems the additional difficulties result from applying controllers with different sampling periods and asynchronous sampling instants. Then, it is actual the question: which methods of description and design should be applied for these systems ?

## 2. CT MODEL OF DT SISO SYSTEMS

Consider the DT single input single output (SISO) system shown in Fig.1a, composed of sampler with sampling period h, zero-order hold (ZOH) and the CT plant described by the transfer function (TF) $G(s)$. The plant can contain a delay. We assume that $G(j\omega) \to 0$ when $\omega \to 0$. The system can be described by the DT TF $\bar{G}(z)$. The example of the plots of the ZOH signals are shown in Fig.1b.

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It can be noticed that if the sampling frequency is sufficiently strongly dumped by the CT plant $G(s)$ then the response of the plant to the averaged signal $\bar{u}^*(t)$ is approximately the same as to the signal $u^*(t)$. Since it is $\bar{u}^*(t) = u(t - h//2)$ then the approximate CT model of the DT system from Fig.1a is described by the TF

$$G_e(s) = e^{\frac{-h}{2}s} G(s) \qquad (1)$$

If the sampling frequency is approximately ten or more times greater than the working frequencies, then taking into account the first term of the exponential function expansion we obtain an other CT model of the DT system

$$G_d(s) = \left(1 - \frac{h}{2}s\right) G(s) \qquad (2)$$

Both the models result from the modification of the primary TF $G(s)$; $G_d(s)$ is modified by the derivative and $G_e(s)$, – by the delay.

[1] The paper was supported by the Science Research Committee grant No 3P40301006

The possibility of using the approximate model (1) was noticed and explained in detalis by Kuzin (1962) and is used among others by Franklin at al. (1994).

Consider the two following closed-loop (CL) systems: first the DT system composed of the plant $\bar{G}(z)$ and the controller $\bar{R}(z)$; second the CT system composed of the plant $G_d(s)$ (or $G_e(s)$) and the controller $R(s)$. Let $\bar{R}(z) = R\left(\frac{2}{h}\frac{z-1}{z+1}\right)$. The CT system well approximates the DT system if the sampling frequency $\omega_s = 2\pi//h$ is approximately more than ten times greater than the bandwidth frequency $\omega^*$ of the CT system (Gessing, 1994, 1995).

Thus, the proposed method of design is based on appropriate using two different approximations, namely the CT approximation of the DT plant and the DT Tustin approximation of the CT controller.

Since the model with derivative is simpler it will be preferred for using in the case of CT plants without delay. In the case of CT plants with delay the use of the model with delay is justified since then we obtain no increase of the design complexity.

The controller $R(s)$ can be determined by using any known method of design. One from the recommended is the method basing on frequency characteristics and $M_p$ criterion. The controller $R(s)$ should be chosen so that $M_p = Max_\omega|(G_d(j\omega)R(j\omega))//(1+G_d(j\omega)R(j\omega))| \approx 1.3$

## 3. MV SYSTEMS WITH DIFFERENT SAMPLING PERIODS

Consider the DT MV systems with m independent DT controllers shown in Fig.2. Let the i-th controller has the sampling interval $h_i = 1,2,...,m$. The sampling instants can be nonsynchronized. Assuming that the sampling intervals $h_i$ are sufficiently small we would like to design the controller algorithms in the form of DT TF $\bar{R}_i(z)$, $i = 1,2,..,m$, so that whole the system is stable and has satisfactory time and frequency responses of particular channels. Assume, that in the proposed control structure the inputs $u_i$ of the plant have been appropriately suited to the outputs $y_i$, $i = 1,2,...,m$.

Let $G_{ij}(s)$ be the plant TF between j-th input $u_j$ and i-th output $y_i$, $i,j = 1,2,...,m$. Using previous considerations we can create the CT model with derivative (or with delay) for the part of the multivariable DT system composed of the CT plant and ZOH-s. We have

$$\begin{aligned} G^d_{ij}(s) &= \left(1-\frac{h_j}{2}s\right)G_{ij}(s), \\ i,j &= 1,2,...,m \end{aligned} \quad (3)$$

$$\begin{aligned} G^e_{ij}(s) &= exp\left(-\frac{h_j}{2}s\right)G_{ij}(s), \\ i,j &= 1,2,...,m \end{aligned} \quad (4)$$

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Thus, for the process of design we can consider the system from Fig.2 in which all the samplers disappear, the j-th $ZOH_j$ is replaced with the TF $(1-sh_j//2)$ (or $\exp(-sh_j//2)$) and the i-th DT controller $\bar{R}_i(z)$ is replaced by the CT controller $R_i(s)$, $i = 1,2,...,m$. First, the CT controllers $R_i(s)$ have to be determined using any known method of design of CT MV systems. Then, the DT controllers $\bar{R}_i(z)$ result from the Tustin approximation of the CT controllers $R_i(s)$. This is described below.

### *3.1. Design algorithm*

On the basis of previous considerations the following algorithm for designing the DT controllers can be proposed.

1) For given TF-s $G_{ij}(s)$ and assumed sampling periods $h_j$ create the modified TF-s $G^d_{ij}(s)$ or $G^e_{ij}(s)$, $i,j = 1,2,...,m$.

2) Using one from the known methods of design of CT MV systems choose the CT controllers $R_i(s)$, $i = 1,2,...,m$ for the CT plant described by modified with the derivative (or with delay) TF-s.

3) Check if the sampling periods $h_j$ are sufficiently small, i.e. if the sampling frequency $\omega_j = 2\pi//h_j$ is approximately more than ten times greater than the appropriately calculated bandwidth frequency $\omega^*_j$ of the j-th loop; if not then decrease the sampling period $h_j$ and repeat the steps 1) and 2).

4) The DT controllers $\bar{R}_j(z)$ result from using the Tustin approximation of the CT controllers $R_j(s)$, $j = 1,2,...,m$, i.e.

$$\bar{R}_j(z) = R_j\left(\frac{2}{h}\frac{z-1}{z+1}\right) \quad (5)$$

The bandwidth frequency $\omega^*_j$ for each j-th loop is calculated with accounting the remaining m-1 loops of the system closed.

**Note.** In order to choose the proper sampling periods $h_j$ assuring the fulfilment of the condition $\omega_j \gtrsim 2\omega^*_j$, first the bandwidth frequencies $\omega^{*'}_j$ can be determined for the CT system with the controllers $R'_j(s)$ designed for the multivariable plant $\mathbf{G}(s) = [G_{ij}(s)]$. Next, we can assume that $\omega^*_j \approx \omega^{*'}_j$ and determine $h_j$.

It should be stressed that in the case of different sampling periods, usually, the TF description of the whole MV system is not possible since the system is then nonstationary. This is not exactly noticed in the literature. The nonexistence of the TF description creates a serious difficulties in designing. Thus, in this case the proposed method of design basing on the CT approach has an essential meaning.

## 4. A METHOD OF DESIGN OF CT MV SYSTEMS

Now, consider the CT MV system shown in Fig.2 in which the samplers and ZOH's disappear and the DT controllers $\bar{R}_j(z)$ are replaced with the CT ones $R_j(s)$. Here, the approach will be proposed which makes if possible to apply a onevariable design method, repeatedly. The approach is based on a repetitive procedure in which the CT controller $R_j(s)$ in the j-th loop is designed for the replacement SISO plant resulting from accounting the remaining loops closed (those which already have designed controllers).

Thus, first the controller $R_1(s)$ is designed for the plant $G_{11}(s)$; the remaining loops are open since they have no designed controllers, yet. Second controller $R_2(s)$ is designed for the 2–nd replacement SISO plant resulting from accounting the first loop closed and the remaining open; tird – for the 3–rd replacement SISO plant with accounting two first loops closed and the remaining open; finally, the m-th controller $R_m(s)$–for the m-th replacement SISO plant resulting from accounting all the remaining loops closed.

Then, we repeat our design procedure starting from the first loop and designing the controller $R_j(s)$ in the j-th loop, $j = 1, 2, ..., m$ for the j–th replacement SISO plant resulting from accounting all the remaining loops closed with the controllers designed in the latter steps. If needed, we repeat the design procedure until the further steps give no improvement of the control.

It should be stressed that the proposed design procedure is usually fastly convergent. This is the result of the fact that the change of the controller $R_j(s)$ in the j-th loop influences mainly the characteristics of the j-th open loop. The characteristics of the remaining open loops are significantly less influenced, since for them the loop with $R_j(s)$ is closed and – less sensitive to parameters change.

## 5. EXAMPLE OF CT SYSTEM

Consider the CT plant described by the matrix TF

$$\mathbf{G}(\mathrm{s}) = \begin{bmatrix} \frac{10}{s(s+1)} & \frac{7}{(s+2)(s+3)} \\ \frac{5}{s(s+1)} & \frac{7}{(s+2)(s+3)} \end{bmatrix} \qquad (6)$$

We would like to design the CT controllers $R_1(s)$ and $R_2(s)$ of the both loops, respectively. In the first step the controller $R_1^{'}(s)$ is designed for the plant $G_{11}(s) = 10//(s(s+1))$. In the second step the controller $R_2^{'}(s)$ is designed for the replacement SISO plant of the second loop described by the formula

$$G_2^*(s) = G_{22} - G_{12}\frac{R_1^{'}}{1 + R_1^{'}G_{11}}G_{21} \qquad (7)$$

We obtain successively

$$R_1^{'}(s) = \frac{0.416s + 1}{0.139s + 1}$$
$$R_2^{'}(s) = 150\frac{0.04s + 1}{0.005s + 1} \qquad (8)$$

In the third step the controller $R_1^{''}(s)$ is designed for the replacement SISO plant of the first loop

$$G_1^*(s) = G_{11} - G_{21}\frac{R_2^{'}}{1 + R_2^{'}G_{22}}G_{12} \qquad (9)$$

We obtain

$$R_1^{''}(s) = 100\frac{0.04s + 1}{0.005s + 1} \qquad (10)$$

In the forth step the controller $R_2^{''}(s)$ for the replacement SISO plant $G_2^{*'}$ (determined by (7) with $R_1^{'}$ replaced by $R_1^{''}$) is designed, which however, gives no improvement of the control.

Thus, the final solution obtained in third design step is determined by

$$R_1(s) = R_1^{''}(s),\ R_2(s) = R_2^{'}(s) \qquad (11)$$

The step responses of the CL system with the designed controllers (11) are shown in Fig.3 and 4.

By the way, in Fig.5 it is shown the influence of the controller $R_1(s)$ parameter change on the frequency characteristics of both loops. It is seen that the characteristic of the first loop is significantly influenced by this change while that of the second loop is influenced only insignificantly.

Of course, when the number of variables increase the formulas determining the replacement SISO plants of particular loops become more complex.

## 6. EXAMPLE OF DT SYSTEM

Consider the DT twovariable system shown in Fig.2 in which the plant matrix TF is determined by (6). We would like to design the DT controllers $\bar{R}_1(z)$ and $\bar{R}_2(z)$. Previously, for the same plant the CT controllers (11) have been designed. In the case of DT system there exists some limitation concerning sampling frequencies, which causes that the DT system is slower than the CT one. Taking this into account, as well as the possibility of using different sampling periods we assume $h_1 = 0.18$ and $h_2 = 0.24$. Then, the model with derivative takes the form

$$\mathbf{G}^{\mathrm{d}}(\mathrm{s}) = \begin{bmatrix} \frac{-0.9s+10}{s(s+1)}, & \frac{-0.84s+7}{(s+2)(s+3)} \\ \frac{-0.45s+5}{s(s+1)}, & \frac{-0.84s+7}{(s+2)(s+3)} \end{bmatrix} \qquad (12)$$

For the plant model (12) the described earlier design procedure has been used. First, the previous solution (11) was modified giving a CT system with several times decreased velocity. Then, using the modified solution in two successive design steps the controllers $R_2(s)$ and $R_1(s)$ for the plant (12) were determined. The appropriate Nichols plots are shown in Fig.6. The controllers take the form

$$R_1(s) = 0.4\frac{0.8s+1}{0.1s+1}, \quad R_2(s) = \frac{2.6}{s} \qquad (13)$$

The bandwidth frequencies of both the loops with appropriate replacement SISO plants are $\omega_1^* = 3.57$ and $\omega_2^* = 1.27$. Since we have $\omega_1 = 2\pi//0.18 = 34.9$ and $\omega_2 = 2\pi//0.24 = 26.1$ then $\omega_1//\omega_1^* = 9.78$ and $\omega_2//\omega_2^* = 20.55$. From the Tustin approximation of (13) we obtain finally

$$\bar{R}_1(z) = \frac{1.873684(z-0.7977528)}{z-0.05263158}$$
$$\bar{R}_2(z) = \frac{0.312(z+1)}{z-1} \qquad (14)$$

In Fig. 7, 8 the step responses of both the loops and in Fig.9,10 the responses resulting from loops interaction for the CT system described by (12) and (13) are compared with that of DT system. Since, for the whole DT system the TF description does not exist the corresponding transients were determined by means of SIMULINK program.

It is seen that both transients of the CT model with derivative and of the DT system are mutually very close.

## 7. FINAL CONCLUSIONS

The proposed method based on the modified plant TF's $G_{ij}^d(s)$ or $G_{ij}^e(s)$ together with the Tustin approximation makes it possible to design the DT MV system practically without knowledge concerning the method of description of DT systems.

The proposed method of design of CT MV systems based on the replacement SISO plants makes it possible to apply the design methods of onevariable systems.

The DT MV systems having samplers with different sampling periods usually can't be described by using DT TF's which creates some essential difficulties in their DT description and design.

In the light of this, the proposed appropriate use of both the models of approximation to the appropriate parts of the system gives as a result a simple and useful method of design.

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