CAUSAL AND NONCAUSAL DISCRETE–TIME TRANSFER FUNCTIONS AND THEIR APPLICATIONS[1]

Ryszard Gessing[2]

Abstract. Causal and Noncausal discrete–time transfer functions for some discrete–time system with zero order hold (ZOH), first order hold (FOH) or ideal sampler are distinguished and described. In the case of ZOH or FOH the discussed notions are essential for the system with a continuous time plant described by the rational proper transfer function with the order of numerator equal to that of the denominator, while in the case of ideal sampler for the plant with the order of numerator less by one from that of the denominator. It is noticed that in the case of ZOH and FOH the causal discrete–time transfer functions have an additional causal mode which increases by one the order of the model. It is shown that for description of the closed–loop systems usually the causal transfer functions are used, however in some cases the part of the system can be described by the noncausal transfer functions.

Introduction

In discrete–time (DT) realization of the control a continuous–time (CT) plant G is usually preceded by the zero order hold (ZOH) and sampler giving the commonly considered system shown in Fig.1a. Here, the case is considered in which the plant is described by the transfer function (TF) $G(s)$. Some times, in the place of ZOH the ideal sampler (IS) or the first order hold (FOH) appears.

The problem considered in the present paper concerns the cases when the appropriate time responses have a discontinuity at a sampling instant. For the systems with ZOH or FOH the problem appears if the step response $y_1(t)$ of the plant $G(s)$ is discontinuous at time $t = 0$, i.e. $y_1(0^+) \neq 0$, where $y_1(t) = \mathcal{L}^{-1}[(1//s) \cdot G(s)]$, $\mathcal{L}^{-1}$ is the symbol of the inverse Laplace transform and $y_1(0^+)$ denotes the right–hand side limit at $t = 0$. In the case of the system with IS the problem appears if $g(0^+) \neq 0$, where $g(t)$ is the impulse response of the plant i.e. $g(t) = \mathcal{L}^{-1}[G(s)]$.

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[2]Politechnika /Sl/aska, Instytut Automatyki, ul. Akademicka 16, 44–101 Gliwice, Poland. fax:+4832371165, email: gessing@ia.gliwice.edu.pl

If in a sampling instant the discontinuity appears there arises the question which kind of the limit right– or left–hand side is accounted by the output sampler 2 ? In connection with this it is reasonable ot propose the distinction between a causal and noncausal transfer functions (TF's), respectively. The distinction is not exactly seen in the literature, however in some items the problem is noticed, e.g. Åström (1984) uses only the causal TF, while Kuzin (1962) obtains the same effect by using a modified TF.

From the other hand, the Program CC (1991) in the command "convert" calculates without any comment only noncausal TF's (for the cases of IS, ZOH and FOH – the options 7, 8 and 9, respectively). In accordance with the author's knowledge also in the toolboxes of MATLAB (1990) the problem of causal DT TF's is not noticed.

The causal TF $H(z)$ of the system with ZOH (Fig.1a) can be calculated from the formula

$$H(z) = (1 - z^{-1})\mathcal{Z}[y_1^*(nh)], \tag{1}$$

where $y_1^*(0h) = 0$, $y_1^*(nh) = y_1(nh)$ for $n > 0$, h is the sampling period, $\mathcal{Z}$ is the symbol of the Z–transform, and $(1 - z^{-1})$ is the inverse of the Z–transform of the step function. The noncausal TF $\bar{H}(z)$ differs only in this that $y_1^*(0h) = y_1(0^+)$. Similarly, one can calculate the causal and noncausal TF's for the systems with IS or FOH.

In the case of the systems with ZOH or FOH the causal TF's have meaning only in the case of the rational functions $G(s)$ with the order of numerator equal to that of the denominator. We can say that such plants $G(s)$ have an uninertial channel. In the case of the systems with IS the problem appears for the case of rational $G(s)$ with the order of numerator less by one from that of the denominator. In the practice, the discussed models result usually form a justified neglection of the transients resulting from relatively small time constants (with respect to the sampling interval). The program CC for these models calculates only the noncausal TF's without any warning. It should be stressed that for these models, in the case of closed–loop systems, usually the causal TF's must be applied. In the light of this the lack of the causal TF's in the program CC must be seen as an oversight.

In the present paper the simple method of calculation of the causal and noncausal TF's in general case of the systems with IS, ZOH and FOH is described. It is also shown that usually the causal TF's are used in applications. The case when the noncausal TF's could be used is also discussed.

The System with Zero Order Hold

Let us consider the system (Fig.1a) with ZOH and the plant described by a rational TF in the form

$$G(s) = \frac{b_0 s^n + b_1 s^{n-1} + ... + b_n}{a_0 s^n + a_1 s^{n-1} + ... + a_n} \tag{2}$$

where $b_0 \neq 0$ and $a_0 \neq 0$. The step response of the plant is shown in Fig.1b and 1c, where additionally the DT transient responses $y_1^*(nh)$ used for derivation of the casual and noncausal TF–s, respectively, are shown.

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Dividing the polynomial of the numerator by that of the denominator of (2) we obtain

$$G(s) = c + G_1(s) \tag{3}$$

where $c = b_0//a_0$, $G_1(s)$ is the strictly proper TF in the form

$$G_1(s) = \frac{b_0' s^{n-1} + b_1' s^{n-2} + ... + b_{n-1}'}{a_0 s^n + a_1 s^{n-1} + ... + a_n} \tag{4}$$

and the numerator of (4) is the polynomial rest resulting from the division.

Since the TF $G_1(s)$ is strictly proper (the order of numerator is less from that of the denominator) the problem of noncausal TF does not appear for it. Let us denote

$$H_1(z) = (1 - z^{-1})\mathcal{Z}\{\mathcal{L}^{-1}[\frac{1}{s}G_1(s)]_{|t=nh}\} \tag{5}$$

The causal TF $H(z)$ corresponding to $G(s)$ can be calculated from the formula

$$H(z) = \frac{c}{z} + H_1(z) \tag{6}$$

where the first term on the right hand side of (6) determines the causal TF corresponding to c.

The transformation of $G(s)$ to the causal TF $H(z)$ determined by the formulas (3) $\div$ (6) is denoted by

$$H(z) = \mathcal{D}_c[G(s)] \tag{7}$$

Of course if is valid for the system with ZOH.

The noncausal TF $\bar{H}(z)$ corresponding to $G(s)$ can be calculated from

$$\bar{H}(z) = c + H_1(z) \tag{8}$$

where the first term on the right hand side of (8) determines the noncausal TF corresponding to c.

From the comparison of the formulas (6) and (8) it results that in the causal TF (6) the additional mode z appears which usually increases by one the order of the TF $H(z)$ in relation to that of $G(s)$ (and $\bar{H}(z)$). The same notice will concern the systems with FOH but not the systems with IS as it will be seen in the further examples.

The transformation of $G(s)$ to the noncausal TF $\bar{H}(z)$, determined by the formulas (3) $\div$ (5) and (8), valid for the system with ZOH is denoted by

$$\bar{H}(z) = \mathcal{D}_{nc}[G(s)] \tag{9}$$

Of course, in the case of continuous step responses at $t = 0$ (i.e. for the strictly proper TF $G(s)$) the application of the both the transformations $\mathcal{D}_c$ and $\mathcal{D}_{nc}$ gives the same result, since then the problem of causability does not appear.

The System with Ideal Sampler

For the systems with IS and a rational TF $G(s)$ (Fig.2) it is reasonable to limit the considerations to the case of $G(s)$ for which the order of numerator is less at least by one from that of the denominator.

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Let $g(t)$ be the impulse response of the plant $G(s)$ i.e. $g(t) = \mathcal{L}^{-1}[G(s)]$. The problem of causal TF appears in this case if the order of numerator of $G(s)$ is less by one from that of the denominator. Then, the impulse response is discontinuous at time $t = 0$, i.e. $g(0^+) \neq 0$. The causal TF $I(z)$ for the system with IS can be calculated from the formula

$$I(z) = \mathcal{Z}[g^*(nh)] \tag{10}$$

where $g^*(0h) = 0$, and $g^*(nh) = g(nh)$ for $n > 0$. The noncausal TF $\bar{I}(z)$ for the system with IS differs only in this that $g^*(0h) = g(0^+)$.

Using (10) and the determination of the transformation $\mathcal{D}_c$ one can notice that the causal TF $I(z)$ for the system with IS can be calculated from the formula

$$I(z) = \mathcal{D}_c[sG(s)]\frac{z}{z-1} \tag{11}$$

Similarly, the noncausal TF $\bar{I}(z)$ for the system with IS can be calculated from the formula

$$\bar{I}(z) = \mathcal{D}_{nc}[sG(s)]\frac{z}{z-1} \tag{12}$$

The System with First Order Hold

For the systems with FOH (Fig.3a) and the rational TF $G(s)$ the problem of the causal DT TF appears in the case of $G(s)$ determined by (2) with an uninertial channel. The step response of the FOH, shown in Fig.3b can be written in the form

$$u^*(t) = \mathbf{1}(t) - \mathbf{1}(t-h) + \frac{1}{h}t\mathbf{1}(t) - \frac{1}{h}(t-h)\mathbf{1}(t-h) \tag{13}$$

where $\mathbf{1}(t) = 0$ for $t \leq 0$ and $\mathbf{1}(t) = 1$ for $t > 0$.

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Let $y_2(t)$ be the response of the plant $G(s)$ to the signal (13). The causal TF $F(z)$ for the system with FOH can be calculated from the formula

$$F(z) = (1 - z^{-1})\mathcal{Z}[y_2^*(nh)] \tag{14}$$

where $y_2^*(0h) = 0$ and $y_2^*(nh) = y_2(nh)$ for $n > 0$. The noncausal TF $\bar{F}(z)$ for the system with FOH differs only in this that $y_2^*(0h) = y_2(0^+)$.

Further on, it will be shown the possibility of calculation of the causal and noncausal TF $F(z)$ and $\bar{F}(z)$ for the system with FOH by using the transformations $\mathcal{D}_c$ and $\mathcal{D}_{nc}$. Let $U^*(s) = \mathcal{L}[u^*(t)]$, $Y_2(s) = \mathcal{L}[y_2(t)]$. From (13) we obtain

$$U^*(s) = (\frac{1}{s} + \frac{1}{hs^2})(1 - e^{-sh}) \tag{15}$$

$$Y_2(s) = \frac{1}{s}G(s)(1 + \frac{1}{hs})(1 - e^{-sh}) \tag{16}$$

Using (14) and the transformations $\mathcal{D}_c$ and $\mathcal{D}_{nc}$ we obtain for the causal TF $F(z)$

$$F(z) = \mathcal{D}_c[G(s)(1 + \frac{1}{hs})](1 - z^{-1}) \tag{17}$$

and for the noncausal TF $\bar{F}(z)$

$$\bar{F}(z) = \mathcal{D}_{nc}[G(s)(1 + \frac{1}{hs})](1 - z^{-1}) \tag{18}$$

When to Apply the Causal and Noncausal Transfer Functions ?

Let us focus our attention on the open–loop system with ZOH shown in Fig.1a. The system can be described either by the causal or by the noncausal TF dependently upon the mutual shift in time of the sampling instants of the output y (sampler 2) and input u (sampler 1). If the sampling instants of the output y are close but somewhat delayed with respect to that of the input u (as in Fig.5a) then the system is described approximately by the noncausal TF, while in the opposite case (as in Fig.5b) – by the causal TF. If both the samplers 1 and 2 work precisely synchronously then we must apply the causal TF. It is the result of the causability principle in accordance with which, first appears sampling, then successively the ZOH response and $G(s)$ response.

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In order to describe the closed–loop (CL) system shown in Fig.4 we must apply the causal TF $H(z)$ and we have

$$Y(z) = \frac{H(z)}{1 + H(z)} W(z) \tag{19}$$

Really, in the case of the CL system the sampling of the input e and the output y is realized physically by only one sampler and the sampling instants of the input e and the output y are synchronous. fgg5.msp6.5cm1cm

Now, there arises the question if there exist some CL systems for description of which the noncausal TF is used. Let us consider the system shown in Fig.6. In this system $C(z)$ determines the DT algorithm of the microprocessor controller in the form of the DT TF; the sampler 1 is related to digital/analog converter (represented also by the ZOH); the sampler 2 corresponds to analog/digital converter; $G(s)$ is the TF of the CT plant. If the mutual shift of the sampling instants of the input u and the output y is such as in Fig.5a then the noncausal TF $\bar{H}(z)$ describes the plant. However in this situation the TF $C(z)$ of the controller must have the order of numerator less at least by one from that of the denominator. It means that the DT controller $C(z)$ will then have one step delay in his output reaction to the input.

It can be noticed that the causal TF $H(z)$ has the order of numerator less by one from that of the denominator, while the noncausal TF $\bar{H}(z)$ has the same order of numerator and denominator. Thus, in the case shown in Fig.5a the TF $\bar{H}(z)$ is noncausal but $C(z)$ can be treated as the causal TF of the microprocessor algorithm. It can be noticed that in the case shown in Fig.5b the system from Fig.6 is described by the causal TF $H(z)$, while the TF $\bar{C}(z)$ then can be noncausal i.e. $\bar{C}(z)$ can have the same order of numerator and denominator.

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The considerations of the present section are valid if the mutual shift of the sampling instants of the output and of the input is very small in relation to the sampling period h. Of course the description of the DT system in the form of the causal or noncausal TF is then approximate.

The case of the Finite Shift of the Sampling Instants

If the shift Δ is such that it should be taken into account then for the case shown in Fig.5b the precise TF $H(z)$ results from usual formulas applied for the system shown in Fig.1a in which $G(s)exp(-s\Delta)$ appears in the place of $G(s)$. In the case shown in Fig.5a the precise TF $H(z)$ is determined by the so called modified TF $H(z,\varepsilon)$, $(\varepsilon = \frac{\Delta}{h})$, discussed in the next section.

If the sampling instants of the input u and output y are precisely synchronous, i.e. $\Delta = 0$, then the causal TF–s $H(z)$ and $C(z)$ must be used. It denotes that then the controller with the TF $C(z)$ having the same order of numerator and denominator can not be used since in each sampling instant no time appears for calculation of the output u after obtaining new value of the input e. It also means that if we would like to use the controller giving instantaneous output reaction to the input (i.e. with the same order of numerator and denominator of $C(z)$) then we should design the sampling instant shift Δ shown in Fig.5b and apply the causal TF $H(z)$. The time Δ is then used for calculation of the output after obtaining new value of input. The latter remark is valid both for the case of proper and strictly proper TF $G(s)$.

Modified Transfer Functions

The modified TF's have been created in order to have the possibility of calculation of the time responses between the sampling instants of time nh (Jury, 1958). Let us consider the system with ZOH shown in Fig.1a. Let $y_1(t)$ be the step response of the plant $G(s)$. Let $y_1(nh+\varepsilon h)$, $0 \leq \varepsilon < 1$, be the DT step response calculated at the instants shifted by $\varepsilon h = \Delta$ in relation to the sampling instants nh (Fig.5a). The modified TF $H(z,\varepsilon)$ for given parameter ε can be calculated from the formula

$$H(z,\varepsilon) = (1 - z^{-1})\mathcal{Z}[y_1(nh + \varepsilon h)] \tag{20}$$

It can be noticed that the problem of the causal TF's appears only for $\varepsilon = 0$. For $\varepsilon > 0$ the appropriate shift $\Delta = \varepsilon h$ shown in Fig.5a exists, owing to this the problem of causability becomes inessential. Thus, for the plant determined by the rational TF $G(s)$ the TF $H(z,\varepsilon)$ is noncausal i.e. has the order of numerator equal to that of the denominator. The latter statement does not concern the plants with delay τ described by the TF $G(s) = G^*(s)exp(-s\tau)$, where $G^*(s)$ is a rational TF $G(s)$. In the latter case the TF $H(z,\varepsilon)$ is causal both for $\varepsilon = 0$ and for such $\varepsilon > 0$

for which $\varepsilon h \leq \tau$.

By the way, there exists mutual dependence between the modified TF $H^*(z,\varepsilon)$ corresponding to $G^*(s)$ and the modified TF $H_\tau(z,\varepsilon)$ corresponding to $G^*(s)exp(-s\tau)$. Let n_1 be the maximal integer for which $n_1 h \leq \tau$. Using Fig.7 one can notice that mutual relations between the functions $H^*(z,\varepsilon)$ and $H_\tau(z,\varepsilon)$ take the form

$$H_\tau(z,\varepsilon) = z^{-n_1} H^*(z, \varepsilon - \frac{\tau}{h} + n_1) \quad for\ \varepsilon\ h > \tau - n_1 h$$
$$H_\tau(z,\varepsilon) = z^{-n_1-1} H^*(z, n_1 + 1 - \frac{\tau}{h} + \varepsilon) \quad for\ \varepsilon h \leq \tau - n_1 h \tag{21}$$

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For the system with IS or FOH we denote the modified TF's by $I(z,\varepsilon)$ and $F(z,\varepsilon)$, respectively. The calculation of these functions can be performed by making appropriate modifications of the formulas (10) and (14) resulting from comparison of (1) and (20).

Comments on the CC and MATLAB Programs

In the Program CC (1991) there exists the command "convert" which calculates the DT TF's corresponding to the CT TF's. The command has 10 options, however from our point of view only some of them are interesting. Unfortunately, the interesting options 7, 8 and 9 calculate only the noncausal TF's $\bar{I}(z)$, $\bar{H}(z)$ and $\bar{F}(z)$, respectively, without any warning. The appropriate causal TF's $I(z)$, $H(z)$ and $F(z)$ are not calculated in this program. In the light of above considerations the lack of the causal TF's in the Program CC can cause an inadequate and faulty description of the system.

Also in the MATLAB (1990) toolboxes the causal DT TF's are not considered exactly. But, in them there exists the possibility of calculation of the DT TF's for the system with ZOH. In the discretization toolbox "$C2D$" the matrix exponential function is utilized, very skilfully. Adding insignificant modification the causal TF's can be also calculated. Using the introduced transformations D_c and D_{nc} and the formulas (11), (12) and (17), (18), both the causal and noncausal DT TF's also for the system with IS and FOH can be calculated. This means that the MATLAB technique used in "$C2D$" toolbox can be also applied for calculation of the causal and noncausal DT TF's for the system with ZOH, IS and FOH.

If we create a program for calculation of the modified TF's for the system with ZOH, IS or FOH, i.e. $H(z,\varepsilon)$, $I(z,\varepsilon)$ or $F(z,\varepsilon)$, respectively, then, only the causal

TF $H(z,0)$, $I(z,0)$ or $F(z,0)$ should be created. The noncausal TF's result from $H(z,\varepsilon)$, $I(z,\varepsilon)$ or $F(z,\varepsilon)$ by substitution in them a very small value of the parameter ε, say $\varepsilon = 0.0001$.

Examples

1. The System with Zero Order Hold

Let in the system shown in Fig.1a. $h = 0.5$ and the plant is described by

$$G(s) = \frac{2s+10}{s+1} = 2 + \frac{8}{s+1} \tag{22}$$

The TF (22) can result e.g. from appropriate simplifying the primary TF in the form

$$G_p(s) = \frac{2s+10}{(Ts+1)(s+1)} \tag{23}$$

If the time constant T is very small in relation to the sampling period h then the transient resulting from the mod $(Ts+1)$ can be neglected giving the simplified model (22).

The causal TF $H(z)$ for $h = 0.5$ and $G(s)$ determined by (22) takes the form

$$H(z) = 2\frac{1}{z} + 8\frac{1-D}{z-D} = \frac{5.1478z - 1.2131}{z(z-0.6065)} \tag{24}$$

(where $D = exp(-0.5) = 0.6065$), while the noncausal TF $\bar{H}(z)$ is

$$\bar{H}(z) = 2 + 8\frac{1-D}{z-D} = \frac{2z+1.9347}{z-0.6065} \tag{25}$$

It is seen that the causal TF (24) has the additional causal mode z which increases the order of the TF by one in comparison to that of noncausal TF (25).

2. The System with First Order Hold

Let in the system from Fig.1a the FOH appears in the place of ZOH, $h = 0.5$, and the plant is determined by (22).

The causal TF $F(z)$ for this case takes the form

$$F(z) = \frac{0.8522z^2 - 6.1306z + 1.2131}{z^2(z-0.6065)} \tag{26}$$

while the noncausal TF $\bar{F}(z)$ is

$$\bar{F}(z) = \frac{2z^2 + 3.6392z - 1.7045}{z(z-0.6065)} \tag{27}$$

From the comparison of the TF's (24), (25) with (26), (27) it is seen that the FOH increases the order of the DT TF by one. The causal TF (26) has additionally the causal mode z which increases the order of the TF by one, too.

3. The System with Ideal Sampler

Now, in the system from Fig.1a the IS appears in the place of ZOH, $h = 0.5$ and the plant is described by

$$G(s) = \frac{1}{s+1} \tag{28}$$

In this case the causal TF is

$$I(z) = \frac{0.6065}{z - 0.6065} \tag{29}$$

while for the noncausal TF we obtain

$$\bar{I}(z) = \frac{z}{z - 0.6065} \tag{30}$$

It is seen that in this case both the causal and noncausal TF–s have the same order equal to that of $G(s)$.

Final Conclusions

The program like CC which calculate the noncausal discrete–time transfer functions without warning can cause in some situations a non correct description of the system. This concerns the continuous–time plants with an uninertial channel in the case of ZOH or FOH, and the continuous–time plants with a rational transfer functions having the order of numerator less by one from that of the denominator – in the case of ideal sampler. This kind of the continuous–time plant models can result from appropriate simplification of the primary models obtained from the neglection of the transients resulting from the relatively small time constants.

It should be stressed that for description of the closed–loop systems usually the causal discrete–time transfer functions should be used. However in the some weakly justified situations described in the paper the part of the closed–loop system can be described by the noncausal discrete–time transfer function.

The property of the causal discrete–time transfer functions in the case of ZOH or FOH is that they have an additional causal mode z which increases the order of the system. The last statement does not concern the systems with ideal sampler.

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