ELLIPSOID METHOD FOR QUANTIFYING THE UNCERTAINTY IN WATER SYSTEM STATE ESTIMATION

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Introduction. For a water distribution system simulation the three main causes of uncertainty are: an inaccurate network model; inaccurate predictions of consumptions; and errors associated with measurements (random and systematic). A smaller contribution to simulation uncertainty comes from the inaccuracy of the mathematical solution techniques and from the precision limits of the computers used. The input uncertainty, from all these sources, is transferred, through the state estimation process, and results in estimates of the operating states that are also uncertain. The precise way in which the input uncertainty affects the accuracies of calculated states is rather complicated, many interrelated factors are involved. The distribution of meters throughout the network, the network's topology and the operating state of the network all play an important role.

Confidence limit analysis - the process of quantifying this uncertainty on state estimates - has implications in many areas of water system operations from distribution system design, through telemetry system design and implementation to real-time control and decision support. This paper reports our experiences with the ellipsoid algorithm that has been adapted to provide confidence limits on state estimates in the nonlinear water system state estimation proces.

Background. Within the broad definition of Identification as "the process of constructing a mathematical model of a system from observations and prior knowledge" [3], the confidence limit analysis of state estimates can be considered to be an identification of the parameter-bounding model derived from the bounded noise measurements. Such interpretation of confidence limit analysis is due to Schweppe [1], who was first to introduce to Identification the concept of unknown-but-bounded errors. In this work the following parameter-bounding model, derived from the bounded-noise measurements is used

$$\mathbf{z}=\mathbf{g}(\mathbf{x})+\nu, \quad |\nu_i| \le |e_i^z|, \quad i=1,..,m \tag{1}$$

where ν is a noise vector for the observations and $\mathbf{e}^z$ is the measurement error vector. Equation (1) is simply saying that the measurement values are inexact and have errors that are unknown but fall within a range bounded by $\mathbf{e}^z$. The aim is to determine the set of all state estimates that are feasible according to (1). An ellipsoidal bounding method was suggested by Schweppe [1] and was further developed by other researchers [3], [4], [5]. An alternative confidence limit analysis technique, based on the linear sensitivity analysis was developed by the author, [2], and was successfuly applied to water system state estimation. The objective of this investigation was to determine whether the ellipsoidal bounding technique will prove suitable for confidence limit analysis in water system state estimation.

The Ellipsoid Method. Mathematically, an ellipsoid, E^t , is a region of space defined as follows:

$$E^t := \{\mathbf{x}\in \mathbf{R}^n : (\mathbf{x}-\mathbf{x^t})P_t^{-1}(\mathbf{x}-\mathbf{x^t})\le 1.0 \} \tag{2}$$

for some $\mathbf{x}^t\in \mathbf{R}^n$ and some symmetric and positive-definite matrix P_t . The aim of the ellipsoid method is to start with a large ellipsoid (usually n-dimensional sphere where n is a dimension of a state space) that contains the whole uncertainty set and then generates a sequence of ellipsoids decreasing in size, leading to one that fits the state uncertainty as tightly as possible. The linearised measurement constraints in the state estimation problem can be written as

$$\mathbf{z}^l - g(\hat{\mathbf{x}})+J\hat{\mathbf{x}} \le J\mathbf{x} \le \mathbf{z}^u - g(\hat{\mathbf{x}})+J\hat{\mathbf{x}} \tag{3}$$

for all $\mathbf{x}$ contained in the state uncertainty set. (3) represents m constraints, bounding $J\mathbf{x}$ from above and below. Each of these is taken in turn and used to modify the current ellipsoid. Processing the t^{th} constraint to update the $t-1^{st}$ ellipsoid involves calculation of an ellipsoid that contains an intersection of the E^{t-1} and the region bounded by the t^{th} constraint hyperplanes. E^t is the ellipsoid $\{\mathbf{x}\in \mathbf{R}^n : (\mathbf{x}-\mathbf{x}^t)P_t^{-1}(\mathbf{x}-\mathbf{x}^t)\leq 1.0\}$, where

$$\mathbf{x}^t = \mathbf{x}^{t-1}+(\rho_t\nu_t/(e_t^z)^2)P'_{t-1}\,\mathbf{a}^t \quad (4)$$

$$P_t = (1+\rho_t-(\rho_t\nu_t/((e_t^z)^2+\rho_t g_t)))P'_{t-1} \quad (5)$$

$$P'_{t-1} = (I-(\rho_t/((e_t^z)^2+\rho_t g_t))P'_{t-1}\,\mathbf{a}^t(\mathbf{a}^t)^T)P_{t-1} \quad (6)$$

$$g_t = (\mathbf{a}^t)^T P_{t-1}\mathbf{a}^t \quad (7)$$

$$\nu_t = 0.5(z_t^u-z_t^l)-(g(\mathbf{x}^t))_t+(J\mathbf{x}^t)_t-(\mathbf{a}^t)^T\mathbf{x}^{t-1} \quad (8)$$

In these equations, P_{t-1} and $\mathbf{x}^{t-1}$ are the positive definite matrix and centre vector, respectively, for the previous ellipsoid, E^{t-1} , and ρ_t can be any non-negative real value. The value, e_t^z used in these equations, is the t^{th} element of the measurement error vector. It should be noted that, despite the fact that (2) refers to P_t^{-1} matrix in its inverted form and (4) to (8) do not, no matrix inversion is involved in the algorithm. In fact, the matrix P_t^{-1} need never be known as all updating is performed using patrices P_{t-1} and P_t. The parameter ρ_t controls the rate of 'shrinking' of the ellipsoids. For ρ_t=0, it is easily seen that E^t is just the same as E^{t-1}. Fogel and Huang [4] suggested the optimisation of the parameter ρ_t by minimising the volume of each new ellipsoid or minimising the sum of squares of ellipsoid semiaxis.

On termination of the algorithm, the confidence limits for each variable are calculated from the final positive-definite matrix, P^t, and the final centre vector $\mathbf{x}^t$ as follows

$$x_i^{1u} = x_i^t + \sqrt{P_t(i,i)} \quad (9)$$

$$x_i^{1l} = x_i^t - \sqrt{P_t(i,i)} \quad (10)$$

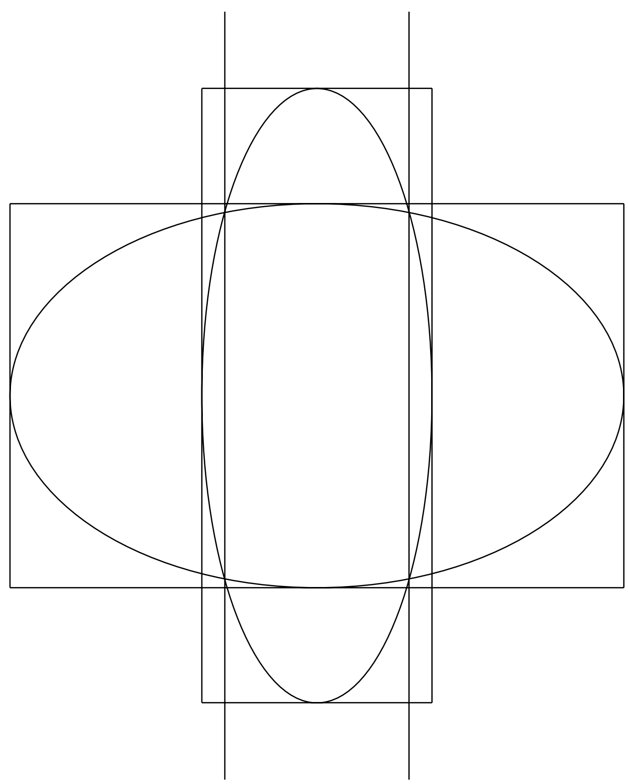

Figure 1 Ellipsoid update increases vertical bounds

Conclusions. An efficient ellipsoid algorithm has been implemented as a general routine for confidence limit analysis in the state estimation process. Extensive tests of the algorithm on randomly generated, dense state estimation problems have shown that the algorithm compares favourably with other linear confidence limit algorithms [2]. These findings are consistent with the results reported by other researchers [3], [5]. However, further tests concerning the state estimation of the water distribution systems, which are described by large sparse matrices, have revealed that the accuracy of bounds on the state uncertainties produced by the ellipsoid algorithm is poor compared to such methods as reported in [2].

The poor performance of the ellipsoid algorithm on the sparse state estimation problem has been attributed to the fact that in such a problem each constraint binds only a few of the state variables. Since in the ellipsoid algorithm, the constraints are considered individually, only few state variables that are explicitly bound by a given constraint, can improve their respective confidence limits. The majority of state variables, which are not bound by a given constraint, are likely to have their confidence limits increased. This is illustrated in Figure 1 using a 2-dimensional orthogonal state space. The large ellipsoid E^t represents a confidence limit set calculated up to the time t. The next iteration of the ellipsoid algorithm processes a pair of vertical hyperplanes which constrain the horizontal state variable but not the vertical one. As a result, the new ellipsoid E^{t+1} has tighter confidence limits than E^t in the horizontal direction but looser ones in the vertical direction. When the ellipsoid algorithm is applied in the context of a large scale sparse state estimation, it is easy to see that without some under-relaxation scheme the majority of the state variables will have their confidence limits looser than required to satisfy the problem constraints. Also, the confidence limits on an idividual state variable depends critically on the order of processing of the constraints.

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